SOVIETICA

PUBLICATIONS

OF THE INSTITUTE OF EAST-EUROPEAN STUDIES

UNIVERSITY OF FRIBOURG / SWITZERLAND

Edited by

J. M. BOCHEŃSKI

TWO SOVIET STUDIES ON FREGE

B. V. BIRJUKOV

TWO SOVIET STUDIES ON FREGE

Translated and edited by

IGNACIO ANGELELLI

D. REIDEL PUBLISHING COMPANY / DORDRECHT-HOLLAND

ISBN-13: 978-94-010-3613-9 e-ISBN-13: 978-94-010-3611-5
DOI: 10.1007/978-94-010-3611-5

© 1964 by D. Reidel Publishing Company, Dordrecht, Holland
No part of this book may be reproduced in any form, by print, photoprint, microfilm, or any other means without permission from the publisher

TABLE OF CONTENTS

INTRODUCTION VII

GLOSSARY

A. Some Fregean terms with their Russian and English translations XIX

B. Some contemporary logico-mathematical terms in Russian and English XX

I. ON FREGE'S WORKS ON PHILOSOPHICAL PROBLEMS OF MATHEMATICS 1

II. THE THEORY OF SENSE OF GOTTLOB FREGE 52

TRANSLITERATION SYSTEM 100

INDEX OF NAMES 101

INTRODUCTION

The significance of the two papers by B.V. Birjukov[1] on Frege within Soviet studies on logic and its history is indicated by G. I. Ruzavin and P. V. Tavanec in their article 'Fundamental Periods in the Evolution of Formal Logic' in the collective volume *Philosophical Questions of Contemporary Formal Logic.*[2] There (page 18) while the organization of "systematic studies on history of logic" is proposed as "one of the fundamental tasks for Marxist logicians", reference is made to a series of recent publications which suggest that such a task is already being accomplished. These are A. S. Axmanov's *The Logical Doctrine of Aristotle*[3], V. F. Asmus' 'Criticism of the Bourgeois Idealist Logical Doctrine in the Era of Imperialism'[4], in *Voprosy Logiki* (Logical Questions), P. S. Popov's *A History of Modern Logic*[5] and B. V. Birjukov's 'G. Frege's Theory of Sense' in the collective work *Applications of Logic in Science and Technology.*[6] In this book, published by the Academy of Sciences of the USSR, Moscow, in a printing of 10 000 copies, Birjukov's article fills 56 pages.

Before this one, however, Birjukov published another study on Frege: 'On Frege's Works on Philosophical Problems of Mathematics' in the collective volume *Philosophical Questions of Natural Sciences*[7], published in a printing of 8 000 copies by the Moscow University Press. This article fills 45 pages.

Despite of a series of criticisms – *some* of which will be stated in this introduction – we are of the opinion that the translation of both papers of Birjukov on Frege makes accessible, for those who do not read Russian, a certainly original contribution to the rapidly increasing Fregean bibliography. It is hardly necessary to point out that the originality lies mostly in the general philosophical background of the author, from which a set of quite unusual objections is addressed to Frege.[8]

The following schema represents Frege's situation in Marxism-Leninism, according to B. V. Birjukov. For the sake of comparison we also include the names of two other outstanding contemporary

philosophers:

	Dialectics	Metaphysics
Materialism		Frege
Objective Idealism		(Frege for A. Kol'man) Russell$_1$
Subjective Idealism		Russell$_2$ Wittgenstein

(For Russell's two successive positions see I, note 7; for Wittgenstein, see II, note 11. For Kol'man's appreciation see I, note 61)

The pointed lines in our schema reflect the following fact: although it is certain that Frege was not a subjective idealist, and although he cannot be condemned as a Platonist, i.e. as an objective idealist, nevertheless, his absolute lack of *dialectics*, i.e. his being still a metaphysician, does not allow one to clear him completely of objective idealism.

We assume the sense of the above terminology according to *Osnovy Marksizma-Leninizma* (Moscow, 1960; 900000 copies). *Materialism–idealism* is an opposition within theory of knowledge. The former acknowledges the existence of a real world, independent of the mind, while the second denies such independence. *Dialectics-metaphysics* is an opposition concerning the acceptance or refusal of movement and evolution. Again, idealism may be either subjective or objective, but it is only *subjective idealism* which opposes itself to materialism in the above-explained sense. *Objective idealism* is not idealism at all in the sense defined above: it postulates a reality independent of us, only that this reality is Plato's and Hegel's *ideal* reality.

The author is quite definite in not considering Frege a subjective idealist. This is essentially right, but should still receive some qualifications. First, the author leaves aside – it would seem deliberately – a very suggestive *nuance* in Frege's anti-idealist struggle.

This nuance is obvious in the original Fregean *contexts* of the fragments quoted and referred to by Birjukov in I, notes 16 and 17, but it does not appear so clearly in the author's commentary. Frege is not saying to idealists: things *are* and *exist* independently of our minds (to use *Osnovy*'s formula) but: the *meaning* of our sentences cannot be reduced to merely concern our own private and subjective representations (*Vorstellungen*); the reason being simply that *we don't mean* this. This sort of intra-linguistic anti-idealism – quite a bit weaker than the Leninist anti-idealist phraseology – is indeed rather trivial once Frege's precise sense of Vorstellung is taken into account. *Vorstellung* means for Frege a particular fact (accident, event) inhering in a particular mind at a particular moment or at several particular moments. It is hard to think that even a subjective idealist would disagree with Frege in saying that we don't speak about such *Vorstellungen*. This is precisely observed by Frege: subjective idealists work, in the final analysis, with an ambiguity concealed by this word. The terminology can be easily changed, as Husserl did.[9] The question is quite trivial and Frege's terminological distinctions themselves are interesting only as referring to the immediate context of nineteenth century philosophy: the same distinction was formally declared to be *vulgaris* three centuries before.[10] Of course, it cannot be said that Frege reduces his anti-idealism to a semantic requirement.[11] This is perhaps only a detail to be taken into account. Frege says many times that his logico-mathematical entities are *objective*. Birjukov makes large use of this. But still another *nuance* seems to have escaped him.

The *Osnovy Marksizma-Leninizma* (1960, p. 15) defines "objective" as "existing outside the human mind and independently of it". It seems as if Birjukov had simply identified Frege's "objectivity" with Lenin's. Perhaps many passages of the Fregean *corpus* could support such identification, but many others tend rather to cast some doubts on it. In normal philosophical terminology "objective" is assigned at least *two* senses. Eisler's *Kant-Lexikon* gives for the term the following meanings: (a) *Den Dingen an sich zukommend*; (b) (*kritisch-idealistisch*) *für alle* (*menschlichen*) *Subjekte gleicherweise gültig und zugleich für die Objekte als Gegenstände möglicher Erfahrung* (*Erscheinungen*) *gültig, obzwar den Dingen an*

sich nicht zukommend . . ..[12] It is interesting to observe that when Frege characterizes objectivity as "being accessible to many", "common property of many", etc. we are reminded rather of the second Kantian sense than of the first. A final decision – if any is possible – could only be taken by means of a *Frege-Lexikon*, which we lack.

But if it is not right to identify – without further ado – Frege's objectivity with that of Lenin, it would be on the contrary possible to adapt Marxist-Leninist terminology to the above-mentioned ambiguity of the word "objective", which is normal in Western philosophy and seems to be the case for Frege. And that without any *tour de force*. The *Osnovy Marksizma-Leninizma* (p. 35) says that "*ideas* and *concepts* don't exist independently but just *in our minds*". "In" has for Boethius at least nine senses; we don't know if this preposition receives as many distinctions in Marxism-Leninism but in the quoted text it cannot mean to be *in* our minds as Fregean *Vorstellungen* are supposed to be, i.e. by a sort of inherence rather similar to how our own hairs are in our head. Obviously, ideas and concepts are *in* our minds – for Marxism-Leninism – in a sense which admits of *some* objectivity. Otherwise the *ideas* and *concepts* mentioned by *Osnovy* would be highly equivocal terms and in each case it should be asked whether the ideas and concepts of John are meant or rather those of Peter (as it is the case for John's and Peter's hairs). That objectivity or quasi-objectivity is easily assimilable to the *weak* sense of the term in the Western philosophical tradition. By such a weak sense we would at least mean all those interpretations for which "objective" does not imply independent existence, without suggesting a too hasty identification of the several weak senses available. Among these, that of the Scholastics is particularly noteworthy, reflected in the phrase: *habere esse objectivum tantum*. The schoolmen – at least the non-Platonists – used to say that universals *qua* universals do not exist independently of us, which is in perfect harmony with *Osnovy Marksizma-Leninizma*. And they added that universals as such have a mere objective being: their being reduces to being "in front of" (*ob*) the subject. The introduction of a term for qualifying the ontological status of ideas and concepts would be, as we see, very useful for Marxism-Leninism,

and it would facilitate a comparison with other philosophies. Then, for our present question, it would be easy to make a parallel between the two senses of Frege's "objectivity" and the two senses of "objectivity" in Marxism-Leninism.

The author is not so categorical for Frege's not being an *objective idealist*. Nevertheless, Birjukov arrives at the conclusion that Frege was not a Platonist, and this in explicit opposition to the normal appreciation. We think that the author is right, but again a mere terminological question is involved. For Marxism-Leninism, Platonism is a species of objective idealism. Objective idealism commits the error of assigning self-existence – in some Platonic heaven – to properties (concepts, ideas). Marxism-Leninism rejects objective idealism but not in the sense of absolutely denying properties: Marxism-Leninism admits a *stratified* reality and a recurring phrase in the theory of knowledge of *Osnovy Marksizma-Leninizma* is "things and properties" (both being reflected by *sensation*: *Oščuščenija . . . éto otraženija, ili obrazy, veščej i ix svojstv*[13], p. 100). *Osnovy Marksizma-Leninizma* insists on the fact that there are properties and that this is precisely an advantage granted to human beings because of the *generality* (with its consequences, like prediction, etc.) which characterizes properties (Chapter on Theory of Knowledge: 'Teorija Poznanija', *passim*). But on the other hand it insists upon the fact that properties are *in* the things, that universality shouldn't become *separation*, namely, objective idealism. It is obvious that Lenin's arguments on this (as quoted and commented by *Osnovy Marksizma-Leninizma* Chapter 1,6: "objective idealism") imply a reiteration of the old distinction *in re, ante rem, post rem*. For Marxism-Leninism, as for most of scholastic philosophers, it is right to assert the existence of properties, but under the supplementary clause: *in* the things. Platonism, thus, only appears when the properties are assigned self-subsistence. On the contrary, for contemporary Western philosophers[14] "Platonism" has a weaker sense, due to the fact that such distinctions as "property in the things" or "property in the mind", etc., are not normally considered. This is necessarily to be taken into account in order to appreciate Birjukov's judgment on Frege's "Platonism". (He could arrive at a similar conclusion by a simpler way: the

Ungesättigkeit of functions, but he has not stressed *this* aspect of functions.)

The foregoing remarks are also necessary to make sense of Birjukov's insistence on showing that Fregean "concepts" are the *common properties* of things. The author tries to prove that Frege's *Begriff* (the most familiar example of *Funktion*) is not considered by the author of *Grundgesetze der Arithmetik* as a "separate entity", a "Platonic Idea", etc., but is closely dependent (by "reflection", "image", etc.) on the properties *in* the things. The author complains that Frege uses the term "concept" in a different sense than the usual (I, p. 163). (Of course, if by "usual" sense of "concept" the author really means Marxist-Leninist terminology – as seems to be the case – then his complaints about Frege's use of the term "concept" are rather anachronistic.)

As expressed in our initial schema, Frege cannot be considered as totally free of objective idealism. The author uses the term "moderate realism" for a final qualification; it should be understood as designating a middleway between objective idealism and full materialism.

The reason for this is Frege's lacking of *dialectics*. Frege was indeed a *metaphysician*.

In relation to dialectics we are told many things in both papers. Some of them are not only meaningless but almost amusing; for example, the implicit suggestion that Frege should not have worried about the contradiction discovered by Russell because that was just a dialectical step in the evolution of logic (I, p. 172). This belongs to a wider accusation made against Western philosophers of mathematics who have spoken of *crisis* in the foundations where in fact there just was the "deep dialectical nature of the most primitive concepts of mathematics and logic" at work (I, p. 134). Other references to dialectics are at least "thinkable" and may even be a stimulating factor of meditation, as the rejection of the universal self-identity (II, note 7). A further class of references to dialectics includes the accusation that Frege was not concerned with movement and evolution (a *resumé* of this in II, p. 551); but this should be less vaguely stated – at least for Western readers not perfectly aware of all the implications of such an accusation. Further,

dialectics is said (I, p. 162) to have given the first sound treatment of the problem of universals. The same reasons as in the previous case make it impossible to seriously treat such a statement. In particular, the reader (the Western reader at least) would like to know what dialectics *adds* to what could be called an *aporetic* treatment of universal-singular relations, such as is obvious throughout Aristotle's *Metaphysics*. The latter is just an impartial statement of thesis and antithesis (book *B*) and a successive consideration of all possibilities (book *Z*). Birjukov's quotations of Lenin seem only to require, in *addition*, a *simultaneous* consideration of all the possibilities – which is a tiring psychological effort rather than objective dialectics. But, of course, we do not have the right of speculating on a brief text: we would only ask that such statements about the priority in the dealing with the problem of universals be better developped.

An essential feature of II is the establishment of an equivalence between *extensionalism* (is the author clear about this?) and *metaphysics*. This seems to be understood in the following way. Frege wanted to preserve his principle of substitution of names having the same reference (*Bedeutung*); but for that he was forced to kill some vital aspects of language; for example the *Beleuchtung*, the peculiar "light" under which concrete fragments of linguistic intercourse appear. All this seems to be reducible to one more attack on "material implication". This, if normal among Western philosophers ignoring logic, is quite surprising in a scholar like Birjukov. It is not necessary to engage in such argument, but we would only observe that if there is a logician who has been careful in translating the horseshoe by "if . . . then", it is precisely Frege. In opposition to *Principia Mathematica*'s rapid decision (Volume I, p. 7), Frege began by not admitting the undiscriminated "*wenn . . . so*" rendering of his conditional sign in the *Begriffsschrift* and in *Grundgesetze* I, § 12 and § 13 seem to indicate that the "if-then" reading only appears with "formal" implication. The short paper '*Über den Zweck der Begriffsschrift*' explicitly forbids the reading "*wenn . . . so*" when material implication is meant. Frege's last paper, 'Gedankengefüge', adopts in fact the "if . . . then" reading without restrictions – as has become customary – but only after serious specula-

tions concerning the legitimacy of such a rendering.

A final appearance of dialectics concerns Frege's understanding of the logical laws. Frege assigned a universal value to these. Without asking too many precisions about this "universality", it constitutes an essential piece of Birjukov's argument.[15] From I, notes 51 and 54, and page 165, we can draw the following schema:

1. Frege understood logical laws as absolutely universal.
2. Frege assumed a universal domain of objects.
3. The class of objects having a common property = the extension of the concept.

The author says that 1 implies 2, and 2 implies (but is not implied by) 3. In particular, the author assigns to the second premiss the responsibility for the rejection of "Schröder's anticipation of the simple theory of types" – to speak with A. Church. This is a question beyond the scope of the present introduction. We would only mention – apart from the well-known paper by A. Church – the recently published study by H. D. Sluga: 'Frege und die Typentheorie' (in *Logik und Logikkalkül*, Verlag K. Alber, München, 1962). Birjukov's argument would perhaps find some support in some statements of H. D. Sluga concerning the law of excluded middle (p. 207).

The third point stated above belongs again to the general frame of showing that "concept" – in this case "extension of a concept" – does not imply objective idealism but, because of understanding "extension of a concept" as a set of *things* having a *common property* (which is the right phraseology for expressing the right conception of universals: *common* but *in re*), Frege is not against Marxist-Leninist presuppositions.

Incidentally it should be observed that *extension* (of concepts) is a controversial term in Soviet logic. In fact, dialecticians affirm that the concept is not a mere set of marks (*Merkmalen*) but implies other complicated relations with the particulars (see e.g. Rozental's *Dialectical Logic*). In this case the traditional laws of reciprocity fail. Other logicians – or better, formal logicians – in the Soviet Union understand extension in the traditional Port-Royal sense. For an interesting *status quaestionis* of all this, see V. T. Pavlov: *Otnošenija meždu ponjatijami* (Relations among Concepts), Kiev Univ. Press, 1961, pp. 14–17.

The author's main argument in I, Section 3, seems to be that both classes and concepts, through the *common properties of things*, are rooted in the material world itself. Because of this, to affirm their objectivity, as Soviet mathematicians do and as Frege did (in Birjukov's view), is *not* Platonism.

As for the speculations on Frege's *Wertverlauf*, it should be *at least* noticed that the author works too much on the idea of *Wertverlauf* as the graphic of a function, i.e. as a set of couples; he says that this idea "follows" from *Funktion und Begriff*, but it does not follow at all. In *Funktion und Begriff* Frege is not quite clear on what really is a *Wertverlauf*, but the graphical representation (or curve) is only said to be an *intuitive image* of it (p. 8) and, on the other hand, the reiterated identification of *Wertverlauf* with *Begriffsumfang* "in the traditional sense", either in *Funktion und Begriff* (pp. 16, 19) or in *Grundgesetze der Arithmetik* (I, § 3, § 9, p. 240), makes it clear that the set of arguments is meant.[16] (For this question in general see W. Kneale and M. Kneale: *The Development of Logic*, Blackwell, Oxford 1962, p. 501; H. Scholz: *Die sogenannte Definitionen durch Abstraktion*, 1935, *Anhang I.*)

As for the references to Fregean semantics it is not necessary to mention such classics as Carnap's *Meaning and Necessity* and Church's developments. Scholz-Hasenjaeger, *Grundzüge der mathematischen Logik* (1960) is also important in this reference. Francesca Rivetti Barbó, *'Il Senso e il Significato di Frege'* in: *Studi di Filosofia e di Storia della Filosofia in onore di F. Olgiati* (Soc. Edit. Vita e Pensiero, Milano, 1961) is a remarkable contribution to the subject. Among the vast amount of minor references, we would mention P. Geach's view, expressed in G. E. M. Anscombe and P. T. Geach, *Three Philosophers* (Blackwell, Oxford, 1961, p. 162).

One could hardly call "Birjukov's mistake" what is an almost general opinion – and could not be otherwise until Fregean *inedita* are published. *Sinn* and *Bedeutung* are not connotation and denotation; given a predicator of degree one, the *Bedeutung* is not the class or extension, but the very abstract or ideal entity designated by the predicator; this abstract entity, as any other entity, may be seen from different points of view, and these make the *Sinn*(*e*). In 1891 Frege wrote a letter to Husserl, where *Sinn* and *Bedeutung* in pre-

dicators is clearly explained by means of schemata (this belongs to the Husserl-Archiv, Université de Louvain). See, on the contrary, Carnap's conjecture in *Meaning and Necessity* § 29. Carnap's diagram (*Meaning and Necessity*, §30) should be corrected as concerns properties. It is easy to see how much of Birjukov's argument depends upon this wrong assumption.[17]

I, p. 158 and p. 173 contain a brief reference to Fregean types. We would point out the fact that it is too meager in a context dealing with the idea of *Funktion* in Frege, and that it is even dangerous for the correct appreciation of "types" in Frege. To say "and so on" in reference to "functions of functions" is equivocal, because it may suggest that Russellian types (and orders) are meant. This is not the place for discussing this interesting chapter of contemporary logic, but it should also be mentioned that the above quoted study of H. D. Sluga contains the assertion that Fregean types are (or may be, because Frege only considered *three*) infinite. It would be necessary to provide examples (construction and interpretation) of Fregean functions of, say, fourth degree.

Birjukov's treatment of *function* seems to elude the point of *Ungesättigkeit* in favour of *Zusammengehörigkeit* (= mapping). The relative incompatibility of both approaches becomes evident if we consider that a mapping can be anything except "unsaturated" or "ambiguous" as *Principia Mathematica* would say. Is the procedure adopted by Birjukov dependent on his basic philosophy?

As minor remarks, we would add the following:

I, p. 152: Frege *does* have *Doppelwertverlauf* (*Grundgesetze* I, § 36).

II, p. 504: *Sinn* described as "act of comprehension" must be assumed as a slip.

II, p. 509: A reference is not always a "sensibly perceptible object" in Frege's ontology. Of course, here the wrong identification of *Bedeutung* = *Gegenstand* is operative.

I, p. 156: "> 2" is not the Fregean name of a concept, but "$(\ ) > 2$", i.e. such that the *Argumentstelle* is included.

I, note 40: It is, of course, abusive to say that Russell did not distinguish *function* and *name* of the function. Cf. *Principia* I, p. 48 note.

II, 505: Birjukov ranges himself among those Frege critics who do not point out the fact that *Sinn* and *Bedeutung*, perhaps in spite

of Frege's own assertion, are perfectly visible in the *Begriffsschrift*.

I, p. 141: Frege actually uses the example "Nessus carried Deianira over the Evenus River" first in the affirmative way, putting it along with a historical (non-poetical) statement on Charlemagne. Frege's point is that the mythological statement keeps being neither true nor false even if we deny it, and only then does Frege give the *negative* form of the phrase.

A series of *Errata* has been corrected. In I, notes 2, 9, 37, 40, 42, 47 contain various mistakes in the quotation references. Note 41 involves a peculiar puzzle which we explain *in loco*. Notes 68 and 70 omit "*Grundgesetze*"; note 70 erroneously says "II" instead of "I".

REFERENCES

1. The transliteration system used in this book is that of the Institute of East European Studies of Fribourg (Switzerland) and is to be found on page 100.
2. 'Osnovy etapy razvitija formal'noj logiki', in *Filosofskie voprosy sovremennoj formal'noj logiki*, Moskva, 1962.
3. *Logičeskie učenie Aristotelja*, Moskva, 1953, 190 str.
4. 'Kritika buržuaznyx idealističeskix učenij logiki époxi imperializma', in *Voprosy logiki* (S.R. P. V. Tavanec) Moskva, 1955, pp. 192–284.
5. *Istorija logiki novogo vremeni*, Moskva, 1960, 262 str.
6. 'Teorija smysla Gotloba Frege' in *Primenenie logiki v nauke i texnike*, Izd. AN SSSR, Moskva, 1960, pp. 502–557.
7. 'O rabotax Frege po filosofskim voprosam matematiki' in *Filosofskie voprosy estestvoznanija*, Izd. MGU, Moskva, 1959, Vol. 2, pp. 134–178.
8. Our own remarks inserted in the translation will be in square brackets. The original pagination is always indicated in the margins. We will normally refer to both papers with the following abbreviation: "I" for "*O rabotax Frege po filosofskim voprosam matematiki*", "II" for "*Teorija Smysla G. Frege*". The number following "I" or "II" will indicate the *page in the original*. I wish to express my thanks to T. J. Blakeley and D. D. Comey for their friendly assistance in the preparation of the manuscript. Also to Basil Blackwell & Mott Ltd. for their kind authorization to reproduce some of Frege's texts from P. Geach and M. Black, *Translations from the Philosophical Writings of Gottlob Frege* (Oxford, 1960).
9. See Dagfinn Føllesdal: *Husserl und Frege*, Oslo, In Kommisjon Hos H. Aschehong & Co. (W. Nygmard), 1958, 60 p. In particular pp. 30–34.
10. We mean Suarez in *Disputationes Metaphysicae* II, *sect.* 1 (Gredos, Madrid, 1960–63). The distinction is there expressed as between *conceptus formalis* and *conceptus objectivus* and their characterization would completely satisfy the most rigorous requirements of Frege. A clear presentation of the same subject-matter is given in G. Jacoby: *Die*

Ansprüche der Logistiker auf die Logik und ihre Geschichtschreibung (Kohlhammer, Stuttgart, 1962), p. 112, a paradoxical book, from another point of view, in as far as it assigns to *Logik*, in opposition to *Logistik*, the task of dealing with identity . . . Frege's *Über Sinn und Bedeutung* is indeed a treatise on identity.

11. "As to the relation of semantics to realism, it should be stressed once more that semantics is not dependent upon realistic principles", E. Beth: *The Foundations of Mathematics*, North-Holland Publ. Co., Amsterdam, 1959, p. 625.
12. For this distinction, see also Lalande's *Vocabulaire Technique et Critique de la Philosophie*. Husserl, in *Erfahrung und Urteil* takes it for granted (§ 10). And it is quite interesting to observe how much Lotze, who had Frege as a pupil, relies on this distinction (Lotze: *Logik, her. u. eing. von G. Misch*, Leipzig, *Verlag von F. Meiner, Philosoph. Bibliothek, Band* 141, 1912. Section A of the first chapter, *Die Lehre vom Begriffe*, contains many references to the above-mentioned distinction. G. Misch mentions Frege's relation to Lotze on page XCII).
13. Sensation (is) a reflection or image of the things and their properties.
14. Of course one should distinguish Platonism in informal presentations or in a state of *explicandum* (in Carnap's sense) from Platonism in rigorous formulation.
15. In *Novye Knigi* (New Books) 1963, 9, p. 3, a new book by Birjukov on the same subject is announced: *Krušenie metafizičeskoj koncepcii universal'nosti predmetnoj oblasti v logike* (The Failure of the Metaphysical Conception of the Universality of the Objective Domain in Logic), *Vyssaja škola*, Moscow, 1963.
16. But neither the graphic representation nor the class of objects (in the traditional sense) belong to the formal presentation of *Wertverlauf* in the *Begriffsschrift*. They are interpretations.
17. For example, II, 543, the text immediately preceding note 50.

GLOSSARY

A. SOME FREGEAN TERMS WITH THEIR RUSSIAN AND ENGLISH TRANSLATIONS

Andeuten[1]	неопределённо указывать	(indefinitely) indicate
Ausdruck	выражение	expression
Bedeutung	значение	reference
gewöhnliche Bedeutung	обычное значение	customary reference
ungerade Bedeutung	косвенное значение	indirect reference
Begriff	понятие	concept
Beleuchtung[2]	окраска	light, colour
Bezeichnen	обозначать	designate
Beziehung	отношение	relation
Definition	определение	definition
Eigenname	собственное имя	proper name
Eigenschaft	свойство	property
das Falsche	ложность	falsehood, the False
Funktion	функция	function
Funktionsname, Funktionszeichen[3]	функциональное имя	functional name, functional sign, name of a function
Gedanke	мысль	proposition, thought
Identität, Gleichheit	равенство, тождество	identity
Merkmal	признак	mark[4]
Name	имя	name
ungerade Rede	косвенное речь	indirect discourse
Satz	предложение	sentence
Behauptungssatz	повествовательное предложение	declarative sentence
Nebensatz	придаточное предложение	subordinate clause
Hauptsatz	главное предложение	main clause

Sinn	смысл	sense [5]
gewöhnlicher Sinn	обычный смысл	customary sense
ungerade Sinn	косвенный смысл	indirect sense
ungesättigt	ненасыщенный	unsaturated
Urteil	суждение	judgment
Veränderliche	переменная	variable
Vorstellung	представление	idea [6]
das Wahre	истинность	truth, the True
Wertverlauf	пробег значений функций	range of values, value-range of a function
Zeichen	знак	sign

B. SOME CONTEMPORARY LOGICO-MATHEMATICAL TERMS

двойка	couple
тройка	triple
объём	extension
логическая функция	logical function
множество	set
класс	class
принцип объёмности	principle of extensionality
объёмный	extensional
отношение называния, именования	name relation
содержательное	1. informal, 2. contentful [7]
принцип взаимозаменяемости	principle of interchangeability

REFERENCES

1. Some signs are *names* (of something "complete": proper names; of something "unsaturated": names of functions). Names designate (*bedeuten* or *bezeichnen*. Other signs "indicate" or, more precisely, "indicate indefinitely" (*unbestimmt andeuten*). The latter, *Andeutung*, is for example the semantics of quantified variables (excepting occurrences *in* the quantifiers, *Grundgesetze* I, p. 32). Now, Frege has still two categories of variables, apart from bound variables: the *lateinische Buchstaben* and the free variables proper. The former are what *Principia Mathematica* reads "any" and they also have a role of *Andeutung* (in respect of functions or objects). The second reduce themselves to "show the places where the completing

sign has to be inserted" (*Grundgesetze* I, § 1, §§17, 19; *Was ist eine Funktion*, etc.). Free variables proper ("ξ" and "ζ" in Frege's use) belong rather to Frege's metalogic than to Frege's *Begriffsschrift* or logic (we follow H. Scholz's terminology in *Die Sogenannte Definitionen durch Abstraktion*, Anhang I). The same may be said for still another class of symbols: Greek capital letters. These are used "as if they were names standing for something although I do not specify a reference for them" (*Grundgesetze* 1, § 5), more or less as "Socrates" or "Sortes" in traditional textbooks.

2. This word occurs in *Über Sinn und Bedeutung*, p. 45.
3. For example in *Was ist eine Funktion*.
4. According to Peirce "mark" is the translation of "*nota*" (Baldwin's *Dictionary of Philosophy and Psychology*, article "mark"). Also Mill gives "nota notae . . ." by means of "mark" (System of Logic, II, II, § 2) "*Nota notae . . .*" was in Kant "*Merkmal von Merkmal . . .*". It is interesting to observe also that a XVIIIth century French translation of Wolff gives "*marque*" for "*Merkmal*". In all this terminology there is a subjective and psychological connotation which caused Hegel's and Joseph's reaction (*Wissenschaft der Logik, Drittes Buch, Vom Begriff im Allgemeinen*, and *Anmerkung* to *B* of *Erstes Kapitel*; Joseph: *An Introduction to Logic*, Oxford, 1916). Of course, Frege's technical use of "*Merkmal*" is free from such implications.
5. *Smysl* is *sense* or *meaning* in a broad sense. For Frege's technical *Sinn* we use "sense".
6. This word must be understood in a purely psychological sense, i.e. as Frege understands "*Vorstellung*". It is a technical term in Frege only in a *negative* way. Frege wants to make it clear that he is *not* concerned with "ideas" but with the objective content of such ideas. All this is an unnecessary complication turning around a long tradition where "*conceptus*" was equivocal (but *technically* equivocal).
7. "*Soderžatel'noe*" is – unfortunately – the untranslatable German "*inhaltlich*". In some passages of our translation we propose "contentful" rather than "informal".

I

ON FREGE'S WORKS ON PHILOSOPHICAL PROBLEMS OF MATHEMATICS[1]

The struggle of the fundamental philosophical tendencies – materialism and idealism – in contemporary mathematics concentrates mostly on its foundations. And this is not fortuitous. Precisely in the domain of the foundations of mathematics appeared difficulties related to the most general and universal notions such as "infinity", "set", "function", "object" and others. At the end of the nineteenth century it seemed to the mathematicians that the problem of the foundations of their science was completely solved. In 1900, H. Poincaré wrote: "Il n'y a plus aujourd'hui en Analyse que des nombres entiers ou des systèmes finis ou infinis de nombres entiers, reliés entre eux par un réseau de relations d'égalité ou d'inegalité. Les mathématiques, comme on l'a dit, se sont arithmétisées . . . . On peut dire qu'aujourd'hui la rigueur absolue est atteinte."[2] But a bit later contradictions were discovered in the theory of sets. It became clear that the problem of the foundations of mathematics could not be considered as solved. Idealist thinkers on mathematics began to speak of the "crisis" of the foundations of their science (Weyl, Brouwer, etc.). But this was in fact not a "crisis" but the discovering of the deep dialectical nature of the most primitive notions of mathematics and logic. So, too, was understood the situation in question by Soviet mathematicians who were dialectical materialists. Another attitude was adopted in the West. In relation to the problems of foundation of mathematics, there arose a series of idealistic philosophical tendencies: logicism, formalism, intuitionism. 135
Discussion about the ways of founding mathematics and the philosophical basis of such foundation is still going on. If we look at the history of these discussions we see that they originated with the prominent German mathematician and logician, G. Frege. His work had a double character. On the one hand, Frege was one of the creators of contemporary mathematical logic and a pioneer in the domain of the foundations of mathematics. He is responsible for the first

formal logico-mathematical system in the history of science, including an important part of arithmetic. Frege was a determined opponent of subjectivism and formalism, and his brilliant argumentation against these currents is *à propos* even today. On the other hand, it is known that his system of foundations of arithmetic was contradictory. Frege did not arrive as far as dialectics; his ideas had a profoundly metaphysical character. After Frege, in the questions of foundation of mathematics, in the West reigned a subjectivistic point of view (Russell). The prominent representative of the so-called logicism, B. Russell, took a step backwards in comparison to Frege. According to the mathematician Kurt Gödel, the system of Frege was more subtle and profound than that of Russell. Considering all that has been said above, the analysis of the opinions of Frege on the philosophical questions of mathematics is of indubitable interest.

I. GOTTLOB FREGE AND HIS WORKS

Gottlob Frege was born in Wismar in the year 1848, studied first at the University of Jena (1869–1871) and afterwards at the University of Göttingen (1871–1873), where in 1873 he obtained his doctorate. In the year 1879 he became professor of mathematics at the University of Jena, where he taught until 1918. In 1918 he retired and died in 1925.

Although one fourth of a century of Frege's life falls in the twentieth century, he must be considered as a thinker of the preceding nineteenth century. And this is not only because the major part of his work was written in the preceding century, but also because certain lines of the evolution of logic and mathematics of the twentieth century, which appeared when he was still alive, were not known to him. However, many of Frege's ideas had an evolution in the twentieth century, and some of the problems provided by the thinker are not yet solved.

Frege wrote several important works. In 1879 in Halle appeared Frege's *Begriffsschrift*, "a Formalized Language of pure Thought Modelled upon the Language of Arithmetic". In this work Frege exposed a system of symbolic logic, created by himself, a new

"perfect" language, as he said, constructed for mathematics. The work of 1879 was an important step in the evolution of mathematical logic: in it this science rose to a higher stage. Four years later ap- 136
peared Frege's second work: *The Foundations of Arithmetic. A logico-mathematical enquiry into the concept of number* (1884). This work was devoted to the problem to which Frege gave practically his entire life: the foundation of arithmetic through logic. At the beginning of the last decade of the nineteenth century, Frege published in different reviews and publications, the articles: 'Function and concept' (an address given in Jena in 1891), 'On sense and reference' (1892), 'On concept and object' (1892), a review of *Philosophie der Arithmetik* of Husserl (1894), 'A critical elucidation of some points in E. Schröder's *Vorlesungen über die Algebra der Logik*' (1895). To the subject-matter of these works is related the article 'What is a function?' (1904).

In 1893 appeared the first volume, and ten years later the second volume of his fundamental work *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet* (Vol. I, Jena, 1893; Vol. II, Jena, 1903).

In the last decade of the nineteenth century, Frege came out with a criticism of some of the principal views of David Hilbert concerning the foundations of geometry (the articles 'Über die Grundlagen der Geometrie', 1903 and 1905). Frege was an opponent to the then nascent formalism in mathematics; because of this he engaged a lively polemic with Korselt and Thomae. [3]

In publishing his *Foundations of Arithmetic* Frege thought that his results would be approved, at least in their basis, by mathematicians who would take the pains of considering his arguments. But there were no such mathematicians. In the preface to the first volume of *Grundgesetze der Arithmetik*, published ten years later, Frege angrily regretted that mathematicians had not responded to his work; that even to such scholars as Dedekind and Kronecker they were unknown. [4] Publishing his basic work, Frege did not count any more on a favourable reception and comprehension of the reader. Mathematicians, he said, coming upon his logical expressions such as "concept", "relation", "judgment", think: *metaphysica sunt, non leguntur*!; in similar way, philosophers, seeing the formulae exclaim: *mathematica sunt, non leguntur*! Frege wrote: "I have got farther

away from the traditional ideas and by so doing have given an appearance of paradox to my views."[5]

137 Frege was not wrong. His works – on the whole – were not understood by his nineteenth century contemporaries. This was due to two circumstances. In the fundamental works of Frege (*Begriffsschrift* and *Grundgesetze der Arithmetik*) was applied a complex and cumbersome symbolism, terrifying the reader and separating him from the author's ideas. Frege could not build a simple and clear system of symbols. The second circumstance was that Frege's articles, written without the apparatus of the *Begriffsschrift* and marked by simplicity, clarity of exposition and expressive language, were scattered in various periodicals, in philosophical journals, and remained little known to the mathematical reader. But the essential was that many ideas of Frege in the field of mathematical logic were ahead of their time.

The importance of Frege's works became more and more clear after the appearance of *Principia Mathematica* of Whitehead and Russell (vol. I-III, 1910–1913). Using Frege's system and Peano's symbolism, and Peirce's and Schroeder's theory of relations, B. Russell and Whitehead elaborated a logical system akin to that of Frege. The authors of *Principia Mathematica* did not hide their debt to Frege. The system of Russell and Whitehead develops the system of Frege in a series of points, e.g. the introduction of the theory of types but, as is stressed by K. Gödel,[6] from the point of view of the elaboration of the proofs, it is inferior to that of Frege.

II. FREGE'S STRUGGLE AGAINST SUBJECTIVISM AND FORMALISM

For a long time, the ideas of Frege influenced the evolution of philosophy of mathematics and mathematical logic, through Russell's works. Books and articles of Frege, published in small editions and scattered in old German journals, were practically inaccessible for the reader, particularly for non-Germans. This circumstance could not but be detrimental to the appreciation of the contribution of Frege to the philosophy of mathematics.

B. Russell, thanks to whose works the ideas of Frege began to be

known to a wide circle of readers, was a subjective idealist.[7] Frege was not such. Moreover, he was, in the full sense of the word, an active enemy of scepticism, subjective idealism and solipsism.

Frege strongly opposed the subjective-idealist "psychologistic" orientation in logic. Taking the example of one of the representatives of this tendency – Benno Erdmann, a contemporary author of a widely read book on formal logic – Frege showed where "the harmful 138
descent into idealism" leads. In Erdmann's works, writes Frege, everything is inclosed in the domain of the subjective, even the real objects are described as ideas (*Vorstellungen*). Frege criticised subjective idealism because, from its point of view, the subject of a judgment is dependent on the process of the idea (*Vorstellung*) and on the man having the idea. He was particularly indignant because Erdmann put numbers on the same level with hallucinations. "What a confusion! . . . Mr. Erdmann stays stuck in a psychological-metaphysical morass."[8]

Frege points out the inconsequence of the representatives of subjectivism. The nature of things, he writes, forbids falling into idealism. Erdmann feels this and does not want to confess that, from his point of view, nothing objective exists. But, shows Frege, the futility of these conclusions is evident. In order to disguise the absurdity of his views, Erdmann uses the word "idea" (*Vorstellung*) in different senses – once on the subjective-psychological level, then with an objective content, as something equally present to all. An analogous operation is made with the words "represented" (*Vorgestelltes*) and "object", which, stresses Frege, are made up to confuse things. First these words are to designate something objective – in opposition to ideas (*Vorstellungen*). But this is only an appearance because in the final analysis they designate one and the same thing. Frege says: "Everywhere here (in Erdmann – B. B.) we see the object, of which I have an idea (*Vorstellung*), confused with this idea and then again the differentiation emerges."[9] But Erdmann's attempts to escape from the muddle into which he fell are vain. The mixture which he offers, says Frege, is a consequence of idealism, a result of the tendency of going against the nature of things.

Reading the Fregean criticism of the subjective idealist Erdmann one cannot help remembering the "Machists", with their sophist

tricks with the "element". Lenin in his work *Materialism and Empirio-Criticism*, showed that the verbal tricks of the "Machists" do not help them to eliminate the contradiction between materialism and idealism. "Either the 'element' is a *sensation*, as all empirio-criticists, Mach, Avenarius, Petzoldt, etc., maintain – in which case your philosophy, gentlemen, is *idealism* vainly seeking to hide the nakedness of its solipsism under the cloak of a more 'objective' terminology; or the 'element' is not a sensation – in which case *absolutely no thought whatever* is attached to the 'new' term; it is
139 merely an empty bauble."[10] Leninist criticism of the Machists is addressed not only to Mach, Avenarius and their disciples, but to all the representatives of subjective idealism, who, being ashamed of its excesses, try to disguise their position by means of every sort of subterfuge. It helps us to correctly appreciate Frege's intervention against the subjectivist Erdmann.

By no subterfuges, shows Frege, can the subjectivists hide the evident contradiction in their views. One of these subterfuges was the use of the expression "as such". Frege wrote on this point: "I take the case that someone wants to convince me that all objects are nothing but images on the retina of my eye. Now then, I say nothing for the moment. But, he goes on to affirm that the tower is bigger than the window through which I think that I see it. To this, then I would say: either the tower and the window are not retinal images in my eye and then the tower could be bigger than the window, or the tower and the window are – as you maintain – images on my retina and then the tower is not bigger but rather smaller than the window. Now he tries to extricate himself from the difficulty with 'as such' and says: the retinal image of the tower as such is, in any case, not bigger than that of the window. At this point I would like almost to jump out of my skin and scream at him: now then, the retinal image of the tower is not at all bigger than that of the window and, if the tower were the retinal image of the tower and the window the retinal image of the window, then the tower would not be bigger than the window and, if your logic teaches you otherwise, of what use is it? This 'as such' is a wonderful discovery for confusionist writers who do not want to say either yes or no."[11]

Frege puts this question to Erdmann directly: is the idea (*Vor-*

stellung) the real object of the judgment? If not, it is necessary to recognize the existence of a world of real things. He himself unconditionally takes this point of view: "If we wish to ever escape from the subjective, we must take knowledge as an activity which does not produce the known but rather grasps that which is already present. This image of grasping is properly suited to explaining the thing. When I grasp a pencil a lot goes on in my body: excitation of nerves, changes in the tension and pressure of muscles, ligaments and bones; changes in blood-circulation. But, the totality of these processes is not the pencil, nor do they produce it. The pencil exists independently of these processes. And it is essential for the grasping that there be something there to be grasped; internal changes alone are 140
not grasping. That which we mentally grasp does also exist independently of this activity, of the ideas (*Vorstellungen*) or their modifications which pertain to or accompany this grasping, and is neither the totality of these processes nor is it produced by them as a part of our mental life."[12]

Frege sharply opposed the psychologistic tendency in logic. He regretted the pernicious, as he himself said, invasion of psychology into logic and that formal logic was fully contaminated by psychologism. In the introduction to the *Grundlagen der Arithmetik*, exposing the principles of his investigation, Frege writes: "Always to separate sharply the psychological from the logical, the subjective from the objective."[13]

Publishing the first volume of *Grundgesetze der Arithmetik*, Frege expressed the hope that his book could help to destroy psychologistic logic.

We already know that the psychologistic logic of Frege's day was inadmissible for him above all because of its subjectivist character. If, noted Frege, instead of things themselves, we only consider their subjective images and ideas (*Vorstellungen*), then every subtle and essential distinction would be banished from logic.

About the psychologistic approach to logic he wrote: "This account makes everything subjective and, if we follow it through to the end, does away with truth."[14] He was disgusted by the fact that for subjective-psychologistic logicians "everything is idea (*Vorstel-*

lung)''. "So all in idealism converges with the greatest logical strictness in solipsism. If one indicated with the word 'Moon' something else – namely one of his ideas (*Vorstellungen*) – just like with the sound 'ow' he expresses his pain, the psychological mode of procedure would be justified, but a dispute on the properties of the moon would be pointless: one man would be perfectly justified in affirming about his moon exactly the contrary of what someone else, with the same right, affirmed about his. If we could not conceive anything else but what is in us, then arguments and mutual understanding would be impossible.''[15] In this case, says Frege,
141 there would be no logic at all, which could be recognized as able to settle conflicts of opinions.

Frege categorically rejected the subjective-idealist theory of judgment as connection of ideas (*Vorstellungen*). Psychologistic logic is on the wrong road when it regards the subject and the predicate of the judgment as ideas in the sense of psychology. If every subject and predicate is only an idea, then it is impossible to reach anything objective.

If subjective-idealists were right, then the sentence "Charlemagne defeated the Saxons'' should be regarded just as the sentence "Nessus did not carry Deianeira over the Evenus River''. But it is not true, says Frege, to consider that, speaking about Charlemagne, I would state anything about my ideas (*Vorstellungen*); this is plainly an idealistic fallacy. In fact, we designate by means of a name a real man and we produce a statement about a real man, who is not dependent on our idea (*Vorstellung*) of him. "In the sentence 'This blade of grass is green' I express nothing about my idea; with the words 'this blade of grass' I do not indicate any of my ideas and, if I did, the sentence would be false. Thereby a second fallacy is introduced: my idea of green is predicated about my idea of the blade of grass.''[16]

In another place Frege says: "To assume that in the sentence 'The Moon is smaller than the Earth' the idea of the Moon is in question, would be flatly to misunderstand the sense. If this is what the speaker wanted, he would use the phrase 'my idea of the Moon'.''[17]

Frege was very right in thinking that in the judgment the state-

ment is related to the object itself and therefore one can talk about its truth or falsity. The sentence "Nessus did not carry Deianeira over the Evenus River" precisely cannot pretend to truth because in it the discourse is not about real objects, since Nessus and Deianeira never existed.

According to Frege, a correct understanding of logical laws is decisive for logic. He was not mistaken when he wrote that logical laws must serve as directives for thought in the reaching of truth. He saw the source of his dispute with members of the psychologistic school in a different conception of truth. He reacted against B. Erdmann, who in the first volume of his *Logik* (1892) identified truth with *general validity*, reducing it to the general agreement of those who state the judgment. "For me it is something objective, independent of he who judges; for psychological logicians it is not that."[18]

In the process of judgment, taught Frege, a proposition is expressed and there is a transition from the proposition to the acknowledgment of its truth. "By a proposition I understand not the subjective performance of thinking but its objective content, which is capable of being 142
the common property of several thinkers."[19] Such are his own words.

Frege was a categorical enemy of radical empiricism, which asserts that only sensibly perceived things exist and which denies the reality of the universal. Speaking against Heine, the author of the work *Elements of Theory of Functions*, Frege wrote: "The widespread inclination to recognize only what can be perceived by the senses as existing is only unfavourable for my book. There is an attempt to deny, or at least to overlook, what cannot be thus perceived. Now the objects of arithmetic, that is to say numbers, are of a kind which cannot be thus perceived. How are we to deal with them? Very simply: the signs used for the numbers are explained to be the numbers themselves. Then in the signs we have something visible, and that is the chief thing. No doubt the signs have altogether different properties from the numbers themselves, but what does that matter? We simply ascribe to them the desired properties by means of what we call definitions. How on earth there can be a definition where there is no question about connexions between sign and thing signified by it is a puzzle."[20]

In these words Frege expressed the whole program of the struggle against formalism in mathematics, which considers arithmetic as some kind of game with symbols; against arbitrariness in mathematics; against the right of the mathematician of "freely creating" the object of his science and of assigning to it the desired properties.

Throughout his life, Frege kept up the struggle against "creative definitions". This struggle was a part of his general struggle against subjectivism.

What is a "creative definition" in mathematics? In Frege's time in mathematics it was generally admitted (and had been for some time) that new numbers (negative, imaginary, irrational) could be introduced with the aid of "postulates" or of constructive (creative) definitions. For example, one postulated the existence of a number x of a new kind, such that $xx = 2$, and one gave to this number the name $\sqrt{2}$. For numbers "created" in such a way, one established rules of operation, in addition to the rules for rational numbers. The mathematicians thought that this was enough for the foundation of the arithmetic of irrational numbers.

Frege was the first to condemn clearly and definitely this ap-
143 proach to things when the mathematician usurps the right of "creating" what he needs. He scoffed at the naive conception of certain mathematicians who thought it possible to invent everything that is desired – even what does not exist, even what is completely unthinkable. "No!" exclaimed Frege, "even the mathematician cannot create things at will, any more than the geographer can; he, too, can only discover what is there and give it a name."[21]

In another place Frege writes: "Here it is of importance to make clear what definition is and what we can reach by means of it. It is, it seems, often credited with a creative power; but really all there is to definition is that something is brought out, precisely limited and given a name. The geographer does not create a sea when he draws border lines and says: The part of the surface of the ocean delimited by these lines, I am going to call the Yellow Sea; and, neither can the mathematician really create anything by his act of definition. Nor can we, by a mere definition, magically give to a thing a property which it has not got, apart from the property of now being

called by whatever name one has given it. But that an oval drawn on paper with pen and ink should acquire by definition the property that, when it is added to one, one is the result, I can only regard as a scientific superstition. One might as well make a lazy pupil diligent by a mere definition."[22]

It is absurd to think that the definition is a guarantee of the existence of the defined. "By means of a definition we can neither create an object with any properties we like, nor magically confer any properties we like on an empty name or symbol."[23]

Criticizing Schröder, who in *Vorlesungen über die Algebra der Logik* defined the identical zero as the name of a domain contained in every domain of a given manifold, and at the same time admitted such expressions as "the symbol 0 is added to the domains of our manifold", etc., Frege wrote: "The zero-sign is an oval figure made, e.g., on paper with printer's ink. Now what more does the definition do? Can it endow this figure with any new property whatsoever? On the contrary, the very most that the figure can get is the property of serving as a sign for the thing that we fix upon as its reference. Or does this figure become a domain contained in every domain of some manifold or other, simply by virtue of my saying it is such a domain? If that were possible, it might also not be hard to make diamonds."[24]

It is impossible, showed Frege, to set up as a definition that which 144
requires a proof or other foundation of its truth. In his first letter to Hilbert he says: "Of late a confusion has crept in in reference to definitions in mathematics and many seem to act according to the rule: That which one cannot rightly prove, is considered as a definition."[25]

Frege stressed that to try to get rid of the burden of proof by recourse to the definition, is a logical hocus-pocus.

Frege's opposition to formalism in mathematics is along the same lines.

Frege distinguished two fundamental directions in the foundations of mathematics: formal and contentful (*soderžatel'noe*). "The former calls numbers figures formed by writing, which are manipulated according to arbitrary rules. In other contexts, these figures can, it is true, be signs which mean something; but the formalist mathematicians abstract completely from this. For contentful arithmetic, these

figures are only signs of their real objects, numerical signs, external tools."[26]

"For us", stressed Frege, "only contentful arithmetic can be taken into consideration."[27]

In the second volume of *Grundgesetze der Arithmetik* Frege submits the formal theory of arithmetic to a detailed examination and criticism. He reacted strongly against the mathematician Thomae, who turned mathematics into something like a game of chess, reduced only to operation with signs according to determined rules.

Thomae considered the signs of numbers as similar to chessmen, and definitions as similar to the rules of the game. Taking such an approach, Frege showed, a sign does not denote anything, and that is all that can be said. "It is true that, in this, one little thing is overlooked; viz. that we express a proposition by "$3^2 + 4^2 = 5^2$",
145 while a position of chessmen does not express anything."[28]

Frege observed that formal arithmetic is not in a position to explain why arithmetical propositions happen to be applicable to reality. If one accepts the formal point of view, then "the gulf between arithmetical formulae and their applications would not be bridged".[29] For the fulfilment of this it would be necessary that arithmetical formulae express a meaning and that its rules find their justification in the meaning of the signs. "Why can no application be made of a configuration of chess pieces? Obviously, because it expresses no proposition. If it did so and every chess move conforming to the rules corresponded to a transition from one proposition to another, applications of chess would also be conceivable. Why can arithmetical equations be applied? Only because they express propositions. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules?"[30] The purpose must be knowledge and by this, stresses Frege, is defined all that is done in mathematics.

III. THE THEORY OF FUNCTION AND CONCEPT

For the evaluation of the philosophical position of one or another mathematician it is very important to determine how he understands

sets. The empirico-nominalistic tendency in philosophy of mathematics rejects the reality of sets (classes) of objects. From the empirico-nominalistic point of view it is not true to say that in reality itself there exist classes of people and animals, sets of white objects, black objects, etc. As if only single concrete individuals with their characteristic notes and properties possessed a real existence, and the existence of sets of people, living things, white objects, etc. were only a fiction. To such opinion usually adhere the representatives of neopositivism.

For Frege such a view was completely inadmissible. The presupposed definition of sets (classes, systems) of his time did not satisfy him above all because to this notion was assigned a subjective character. Mentioning Dedekind's definition ("It very frequently happens that for some reason different things *a*, *b*, *c* . . . can be considered from a common point of view; . . . can be put together in the mind; and we then say that they form a *system S*"), Frege wrote that this union and comparison of objects in the mind is not an objective trait. "In whose mind, may I ask? If they are put together in 146
one mind and not in another, do they then form a system? What is to be put together in my mind must doubtless be in my mind. Then do things outside myself not form systems? Is a system a subjective formation in each single mind? Is then the constellation Orion a system? And what are its elements? the stars, the molecules or the atoms?" [31]

Thus we see that Frege unreservedly acknowledged the objectivity of sets. It is necessary still to clarify *what* gives to a given object real existence as a class, as a set. Frege understood and thought that this question on how the separation, the delimitation of classes comes about is very important, and cannot be avoided in logic. How did Frege himself solve this question? Let him speak; he wrote: "Classes are determined by the properties which their individuals must have." [32]

It is necessary to give special attention to the Fregean understanding of *property*. Frege identified the property which is inherent in the individuals of any group with the concept according to which any object is assigned to this group.[33] For example, he wrote: "In fact, what else is there to constitute a class, if we ignore the concepts, the common properties!" [34]

In the preface to the first volume of *Grundgesetze der Arithmetik* Frege says that a statement about the number of objects of a group presupposes that the latter "is always defined by a concept, i.e. by the properties which an object must have in order to enter the group."[35] Referring, in the same preface, to the definition of a concept, Frege writes that it must consist in showing which property an object must possess in order to fall under the given concept. For example, let there be the following definition of zero: "that which added to one gives one". For Frege, the content of this definition consists in showing the property which an object must have in order to fall under the concept of zero.

The Fregean identification of concepts with common properties of objects becomes especially clear when he introduces the distinction
147 between *properties of objects* and *marks of concepts*. The words "mark" and "property" serve Frege for denoting the relations expressed by such sentences as "ϕ is a property inherent in Γ" and "ϕ is a mark inherent to Ω". Frege writes: "I call the concepts under which falls an object, its properties, so that

'to be ϕ is a property of Γ'

is only another way of saying

'Γ falls under the concept ϕ'.

If the object Γ has the properties ϕ, χ and ψ, I may combine them into Ω so that it is the same thing if I say that Γ has the property Ω or that Γ has the properties ϕ, χ and ψ. I then call ϕ, χ and ψ marks of the concept Ω, and at the same time properties of Γ."[36]

Frege offers the following example. Instead of saying:

"2 is a positive number",

"2 is a whole number", and

"2 is less than 10",

we may say:

"2 is a positive whole number less than 10".

Here the predicates

to be a positive number,

to be a whole number, and

to be less than 10, appear as properties of the object 2 and at the same time as marks of the concept

positive whole number smaller than 10.

Thus Frege calls the concept under which an object falls, its property. ϕ, χ and ψ are properties of the object Γ and concepts under which falls Γ. The union of the properties ϕ, χ and ψ may, too, be considered as a property (the property Ω). At the same time, Ω is a concept under which falls Γ and which is made up of component concepts – the marks ϕ, χ and ψ. The complex concept Ω is subordinated to each of the concept-marks.

Let us quote, finally, a passage from the letter of Frege to Liebmann: "Concepts are usually composed of component concepts, marks. *Black silk cloth* has the marks *black*, *silk* and *cloth*. An object which falls under this concept has these marks as properties. That which is *mark* in reference to the concept is *property* in reference to the object falling under the concept."[37]

Thus the words "common property ϕ" and "the concept ϕ" refer 148
essentially to one and the same thing. The difference between them consists only in the way they are used. If we want to use the word "property" we say:

"Γ has the property ϕ".

If we want to apply the word "concept", we must say, according to Frege:

"Γ falls under the concept ϕ".

Finally, we may express the same sense, simply saying:

"Γ is ϕ".

The question arises: how should Frege's identification of concepts and properties be understood? To have an answer to this is the more important as the theory of concept plays a very essential role in his logic (he built his logical calculus as *Begriffsschrift*).

In the article *Begriff und Gegenstand*, replying to Kerry, who had criticized his theory of concept, Frege wrote: "My explanation is not meant as a proper definition. One cannot require that everything shall be defined, any more than one can require that a chemist shall decompose every substance. What is simple cannot be decomposed, and what is logically simple is no more given us at the outset than most of the chemical elements are; it is reached only by means of scientific work. If something has been discovered that is simple, or at least must count as simple for the time being, we shall

have to coin a term for it, since language will not originally contain an expression that exactly corresponds. On the introduction of a name for something logically simple, a definition is not possible; there is nothing for it but to lead the reader or hearer, by means of hints, to understand the words as intended."[38] For Frege, such "logically simple" are what he designates with the words "object", "function", "concept". Because, as we shall see below, concepts for Frege are a particular case of functions, we begin the consideration of Frege's theory of concept by his theory of function.

What is a function? Frege says that, still in the time of the birth of mathematical analysis, the answer to this question would have been: "Function of x is a mathematical expression, containing x, i.e. a formula containing the letter x". "This answer cannot satisfy us, for there no distinction is made between form and content, sign and thing signified."[39] The fundamental principle of the logical method of Frege is the clear distinction between the name and the object designated by this name, the language and the objective
149 content expressed by this language. The methodological value of such an approach is evident. It is precisely the application of this principle – very emphatically and consequently – which puts Frege on a level above not only that of his contemporaries – mathematicians – but also that of many scholars of the subsequent period. In particular B. Russell, in this respect, is inferior to Frege.[40]

Frege distinguishes the function and its name. The object of mathematics are functions themselves, and not their names. Frege stresses that "the analytic expression, as a group of signs, does not at all belong to arithmetic. I think that I can consider the formalist theory, which makes signs the object of this science, completely refuted by my criticism."[41]

Under "function" Frege means that thing whose essence consists in the association of something to something, constructed according to a definite law. If this something is called an object, then the essence of the function will consist in the law according to which to one object another object is put in correspondence. Let us conceive any numerical function, for example sin x. Its essence consists in the establishment of a correspondence between the numbers, whose

names take the place of "x" (arguments) and numbers occurring as values of the whole expression (the values of the function). The law of this correspondence between numbers, for Frege, is expressed by the equation "$y = \sin x$". The principal part of this equation is the sign "sin". This sign must be properly considered as the designation of the essence of the function *sinus*, because it shows the peculiarity of a given law of correspondence. Therefore, it may be regarded as the name of the function.

Names (signs) of functions, for Frege, have one characteristic trait distinguishing them from the names of objects: they are "not saturated", "need to be complemented" by the name of an object. For example, the sign "sin" requires the complement of the symbol of a number: "sin 0", "sin 1", "sin 2". After such completion the name of a function becomes the name of an object (for example, the symbol of a number). The necessity of completion for the sign of a function may be made apparent by using empty parentheses, e.g. "sin ( )". Instead of empty parentheses it is possible to agree to use any sign, for example the letter ξ. Then the sign of the function sinus will take the form "sin ξ", where "ξ" just indicates the place in which it is possible to introduce the completing sign. 150

The symbol "ξ" is what today is usually called a free variable. The explanation of the concept of variable, as it is used in mathematical logic, is an important contribution of Frege. Above all Frege criticized the traditional conception of variable magnitudes. He did not deny variable magnitudes (e.g. magnitudes of length, angles, mass) but considered that they did not concern the domain of mathematics itself but the domain of its applications. Frege stressed that the question of variable numbers has no sense. Every number preserves unchanged its properties. There are no variable numbers; numbers are always constant. It could be conceived, writes Frege, that the expression "number indicating the length of this rod in millimeters" designates a variable number. But it is not so. The given expression is analogous to the expression "the king of this country". Because at different times in a given country there may be different kings, this expression, without indication of time, does not designate any concrete man. "Similarly, the expression 'the number that gives the length of this rod in millimeters' does not designate any number

at all if the time is not mentioned. If mention of a time is added, a number may be thus designated, e.g. 1000; but then this is invariable. If a different time is mentioned, we get a different expression, which may thus also designate a different number, say 1001."[42] It is absurd to imagine that the number 1000 increases and becomes 1001. What actually increases is the length of the rod, and the number expressing its previous length is replaced by another number, the expression of its new length.

What then is the purpose of applying in mathematics such letters as "n", "x", "y", etc., if they are not designations of variable numbers? They generally serve, says Frege, for the expression of universality. When we write "$x^2 + 3x = x(x + 3)$" we want to express the fact that by the replacement of "x" by any designation of a number a true equality will be obtained. The letter "x" plays another role when we have to deal with analytical expressions like "$x^2 + 3x$". Here "x" serves for showing the need of completion, inherent to the name of the function of addition or – which is the same – for marking the empty (argument) places in the name of the function, into which it is possible to introduce the completing signs, the names of objects. The variable does not designate these objects like a name designates an object, but, as Frege says, indicates
151 indefinitely, alludes to them. Speaking in terms of contemporary mathematical logic, the letter "x" in the first case is a bound variable (it is bound by the universal quantifier "for every x"), and in the second case, free. For Frege, it is better not to use the word variable at all, because of its ambiguity and in order not to have a confusion of free variables with names of objects and bound variables, to use as free variables always the same letter ξ (if it is a function of one argument) and letters ξ and ζ (if it is a function of two arguments).

"The peculiarity of functional signs, which we here called 'unsaturatedness', naturally has something answering to it in the functions themselves. They, too, may be called 'unsaturated', and in this way we mark them out as fundamentally different from numbers. Of course, this is not a definition; but likewise none is here possible."[43] "Unsaturatedness" or "incompleteness" of functions themselves consists in the fact that if the function is completed

by an argument (putting its name in the empty place in the name of the function), then we get an object – the value of the function. In distinction to the names of the functions, the names of objects (proper names) are "saturated", i.e. do not require any completion, as the objects themselves are "saturated". Therein lies the fundamental difference between objects and functions, between proper names and signs of functions.

We see that the notion of *object* is closely connected by Frege with the notion of *function*. Frege calls *object* everything that is not a function and that may occur as argument and value of the functions. Among the objects, for Frege, are to be found, above all, the sensibly perceived things, objectively existing outside of us. Apart from them, Frege includes among objects the so-called *logical objects*, for example *numbers*. *Truth, falsehood* and the *value-ranges of functions* are also *logical objects*. All objects, for Frege, make up one *universal objective domain*.

The notion of value-range of a function is very important in the system of Frege. It is introduced with help of the principle of abstraction.

"I use the words", writes Frege, "'the function $\phi(\xi)$ has the same range of values as the function $\psi(\xi)$', to stand for the same thing as the words 'the functions $\phi(\xi)$ and $\psi(\xi)$ have the same values for the same arguments'."[44]

In this way, the range of value is defined as that which two functions have in common if their values are always the same when their 152
arguments are the same.

Frege does not give a direct definition of the value-range of a function.[45] But it is evident that the range of value of a given function is what is usually called the graphic of the function, i.e. a certain set of couples in each of which the first element is an argument and the second the value of the function for this argument (having in mind functions of one argument; for functions of more than one argument, the notion of range of value is not defined by Frege). Such understanding of the notion of range follows, e.g., from the explanation of it which Frege gives in the paper 'Funktion und Begriff'.[46] The introduction of the ranges may be put into symbolic language (we combine Frege's symbolism with contemporary sym-

bolism) in the form of the following important principle of his system:

$$\forall \mathfrak{a}\,(\phi(\mathfrak{a}) = \psi(\mathfrak{a}))\ \text{equiv.}\ \grave{\varepsilon}\,\phi\,(\varepsilon) = \grave{\alpha}\psi(\alpha)$$

Here "$\forall\mathfrak{a}$" is the universal quantifier, and "$\grave{\varepsilon}\phi(\varepsilon)$" and "$\grave{\alpha}\psi(\alpha)$" are the designations, introduced by Frege, of the value-ranges of functions. This equivalence may be read thus: the universality of identity of values of two given functions is equivalent to the identity of what Frege calls the value-ranges of these functions. "The possibility of regarding the identity universally holding between values of functions as a (particular) identity, viz. an identity between ranges of values, is, I think, indemonstrable; it must be taken as a fundamental law of logic."[47] Although the functions themselves are not objects, their ranges of value must be regarded as objects. This follows from the above mentioned principle, in which the expression "$\grave{\varepsilon}\phi(\varepsilon) = \grave{\alpha}\psi(\alpha)$" designates identity of *objects* (because "$\grave{\varepsilon}\phi(\varepsilon)$" and "$\grave{\alpha}\psi(\alpha)$" are "saturated" as far as they do not have argument-places).

The important step forward made by Frege in the understanding of functions, was the admission as arguments and values of functions of every possible object, and not only numbers. This gave him the possibility of regarding concepts as particular cases of functions. The expressions of the form "$\xi^2 = 4$" or "$\xi > 2$" must, too, be regarded as names of functions, says Frege. Thus, the expression "$\xi^2 = 4$" is the name of a function which correlates to the number 1 the False (because the proposition "$1^2 = 4$" is false), to the number 2 the True (because the proposition "$2^2 = 4$" is true), to the number 3 the False, etc. Sentences, resulting from the introduction of names
153 of arguments in the "empty place" of the name of function "$\xi^2=4$" are names of a particular kind, names either of *truth* or of *falsehood*; truth and falsehood, occurring as values of sentences, are called by Frege *truth values* (*Wahrheitswerte*). He writes: "the value of the function $\xi^2 = 4$ is the *truth-value* either of what is true or of what is false."[48] Therefore, truth and falsehood appear as special logical objects.

In contemporary mathematical logic functions similar to the functions "$\xi^2 = 4$" and "$\xi > 2$" are called *logical functions*, (they are also called propositional functions). Frege calls such functions

concepts, if they are functions of one argument, and *relations* if they are of two arguments. According to him, functions of one argument, having as their value truth or falsehood, are the same as that which in logic is called a concept. Therefore "it seems suitable to say that a *concept* is a function whose value is always a truth-value."[49] The range of values of such a function receives from him the name "extension of the concept". The extension of the concepts may also be defined with help of the principle of abstraction, as that which two concepts have in common (functions of one argument), which for the same argument have the same truth-value. As example of such functions Frege proposes $\xi^2 = 1$ and $(\xi + 1)^2 = 2\ (\xi + 1)$. Both functions have the same range of values:

$$\grave{\varepsilon}(\varepsilon^2 = 1) = \grave{\alpha}((\alpha + 1)^2 = 2(\alpha + 1)).$$

"In logic", writes Frege, "this is called identity of the extension of the concepts. Hence we can designate as extension of a concept the value-range of a function whose value for every argument is a truth-value."[50]

In relation to concepts Frege establishes the following way of speaking: if Γ is an object and $\phi(\xi)$ a concept and $\phi(\Gamma)$ is true, then Frege proposes to say: "the object Γ falls under the concept ϕ". For example, the number 2 falls under the concept "$\xi^2 = 4$" because "$2^2 = 4$" is a true sentence. As we already know, for Frege this proposition can also be expressed in terms of properties, saying: "to be a number, which being multiplied by itself, gives the number 4, is a property inherent to the number 2". Finally, it can simply be said: "2 is a number which, if multiplied by itself, gives a number identical to 4".

Logical functions of *two* arguments, whose values may only be truth or falsehood, are called by Frege *relations*. Examples of rela- 154
tions may be: $\xi = \zeta$ (the relation of *identity*) and $\xi > \zeta$ (the relation of *being greater than*). Let there be two objects Γ and Δ and the logical function $\psi(\xi, \zeta)$. If, says Frege, $\psi(\Gamma, \Delta)$ is true (i.e. designates the truth-value "truth"), then we may read it: "the object Γ stands in the relation $\psi(\xi, \zeta)$ to the object Δ".

For Frege, every function, including concepts, has one and the same domain of arguments: the universal objective domain, em-

bracing all the objects of the world. From this arises the possibility of interpreting the extension of a concept as the class of objects possessing a common property.[51] In this, the Fregean notion of extension of a concept coincides with the usual notion of a class, as that which consists (or may consist) of objects. Establishing the equivalence of the expressions

"Γ falls under the concept $\phi(\xi)$",
"to be $\phi(\xi)$ is a property inhering in Γ"

and

"Γ is ϕ",

Frege, most probably, started from the evidence that the meaning of the notion of property, which is introduced in this way, fully coincides with the usual contentful notion of common properties of objects. In any case the coincidence of the Fregean notion of property with that which is usually associated with the words "common property of objects" is not difficult to see.[52]

In fact, let us consider the two truth-functions of one argument: $\xi > 2$ and $\xi > (4 - 2)$. Since, for every object occurring as their argument, both functions have always the same truth-value, they have the same range of values. What is it? Let us suppose that we can make infinite tables for both functions expressing the correspondence of the value of the functions to their arguments. Doing this, we observe that the two tables coincide:

ξ	$\xi > 2$ $\xi > (4 - 2)$
1	false
2	false
3	true
4	true
...	...

155 (For simplicity we begin the column for arguments writing down the designations of natural numbers, but in it must be included, for Frege, not only all numbers but in general every object of the world – more exactly, their names.) As a result, this table is what our two functions have in common, and this means that according

to the definitions we must call it "the range of values" of both functions. For Frege our functions may also be called *concepts* and, instead of the expressions "function $\xi > 2$", "function $\xi > (4 - 2)$", we may say "concept $\xi > 2$", "concept $\xi > (4 - 2)$" or "concept of being greater than 2", "concept of being greater than $(4 - 2)$", and instead of the expressions "range of values of the function $\xi > 2$", "range of values of the function $\xi > (4 - 2)$", we may say "extension of the concept of *being greater than 2*", "extension of the concept of *being greater than* $(4 - 2)$". Our table is the totality of couples (1, false), (2, false), (3, true), (4, true) . . . . This totality of couples is for Frege the extension of the concepts $\xi > 2$ and $\xi > (4 - 2)$. In general, for any two coextensive concepts, that common feature which makes them equivalent, is the fact that to each of them corresponds one and the same set of couples such that the first element of the couple is an object of the objective domain, and the second is either truth or falsehood; *all* couples which differ in their first elements belong to the set and it is not possible that two couples differ *only* in their second elements. It is clear that to every such set of couples corresponds a set of first elements of these couples (i.e. a set of objects of the objective domain), consisting of the elements for which in the couple stands "true". Conversely, to every set of objects of the objective domain (including empty ones, too) corresponds a set of couples of the kind in question, having the following construction: in it enter all the couples, whose first elements consist of objects of the given set, and the second element for all of them is "true"; and also all couples as the first element of which occur objects of the objective domain, which did not enter into the initial set, and whose second element is "false". If every concept is defined in the same objective domain (and for Frege this is granted, since for him the objective domain is universal), then the converse relation is also univocal. Thus, for Frege, to every set of couples of the kind in question ("tables") there is in one-one correspondence a class of objects of the universal objective domain. This class of objects may be also considered, therefore, as that which two coextensive concepts have in common and may be called the "extension" of these concepts.

In our example the extension of the concepts $\xi > 2$ and $\xi > (4 - 2)$ may be presented as a class of objects: 3, 4, 5. . ..[53] For the

156 Fregean definition, each of these objects falls under both concepts, $\xi > 2$ and $\xi > (4 - 2)$, because the propositions

$$\begin{array}{ll} 3 > 2 & 3 > (4 - 2) \\ 4 > 2 & 4 > (4 - 2) \\ 5 > 2 & 5 > (4 - 2) \\ \ldots & \ldots \end{array}$$

are true.

Considering the sentences of the left column (the same analysis may be repeated for the right column) we see that all of them have one and the same component part: "> 2", which it is natural to consider as the expression of a *common property*, inhering in the objects of a given class. We know that this very expression "> 2" for Frege designates a concept. From which it results that the *common property* "greater than 2" and the concept "greater than 2" are plainly identical. Then the membership of an object in the class of objects corresponding to the concept is identical to the presence in the object of the corresponding property and also to the belonging of the object to the extension of the concept, considered as the class of objects falling under the given concept, i.e. having the corresponding common property. Thus the Fregean understanding of a class as the extension of a concept completely coincides with the informal or contentful (*soderžatel'noe*) understanding of class and property. The thesis – which later became essential – that two classes are the same if their elements are the same, is also justified in his theory. Although this is connected by Frege with the acknowledgment of a universal objective domain, nevertheless this thesis is also maintained in the case of a restriction of the domain, on the condition that we limit ourselves to the consideration of classes precisely in this objective domain.[54]

Frege's theory explains well the legitimacy of introducing in mathematics and logic "empty concepts", i.e. concepts with empty extensions. If, as Frege defines, concepts are that whose essence consists in relating truth-values to objects of the objective domain, then concepts which to every object relate the truth-value "false" are not less legitimate than any other. Although the extension of such concepts would not contain any object (the corresponding

class happens to be empty), nevertheless they are completely justified from a logical point of view. If proper names not designating any object are not legitimate in science, it is impossible to get along without empty concepts, Frege rightly stresses.

In Frege's theory concepts are a particular case of functions. We 157
know that, as in the case of functions themselves, for Frege the names of the functions are also "unsaturated", "incompleted". In the case of concepts, their incompleteness is due to their *predicative nature*. In the sentence "the morning star is a planet", the grammatical subject designates an object, and the grammatical predicate a concept (property). The concept-word "planet" is not complete, it has to be completed by the name of the object, in the sense that without such a completion it indicates neither truth nor falsehood. On the contrary, after the completion by means of the name "the morning star", achieved with the help of the copula "is" (which indicates that the given expression is a sentence), it becomes the expression of a proposition and pretends to *truth*.

In distinction to the concept, the object is not predicable. It cannot be said about anything. Frege writes: "The essence of the concept can be indicated by the fact that one says it has a predicative nature. An object can never be said of something. When I say: 'The evening star is Venus', I do not predicate Venus but *to be identical with Venus*. Linguistically, proper names correspond to objects and concept-words (*nomina appellativa*) to the concepts."[55]

In every sentence at least one part must be incomplete, otherwise no connection will be possible among the parts of the proposition. Frege says: "For not all the parts of a proposition can be complete; at least one must be 'unsaturated', or predicative; otherwise they would not hold together."[56]

Thus, in the sentence "2 is a natural number" the expression "natural number" is unsaturated. This sentence expresses the falling of the object 2 under the concept *natural number*. It is necessary to distinguish the relation of an object falling under a concept from the relation between the extension of one concept and the extension of another concept when the first is *subordinated* to the second. Such a relation appears for example in the sentence "all mammals

have red blood''. In this sentence there are *two* parts requiring completion (two concept-words): ''mammals'' and ''having red blood''. The sense of this sentence may be given as follows: ''if something is a mammal then it has red blood'' (the completion of both predicates here is done not by means of a proper name but, as Frege says, by the indefinitely indicating word ''something'', serving for the expression of universality).

158 These theses of Frege's theory have doubtless a rational sense. Frege is right when he shows that the connection between propositions requires the presence in the structure of the sentence of expressions of predicative character. His way of formalizing universal judgments is today usual in symbolic logic. Apart from this it is necessary to remember that within the framework of this theory Frege introduced concepts of second, third and so on degrees, and showed the relations among them. This gave him the possibility of defining number and, too, of throwing light on some questions connected with the notion of existence in logic.

In the understanding of functions and classes, Frege took a new and original path, different from that taken by his predecessors and contemporaries. From Dirichlet[57] on, the notion of function as correspondence between the elements of two classes was established in mathematics. According to this, the notion of class was considered as in some sense previous to the notion of function. At that time mathematicians did not yet know the difficulties which are connected with the notion of class. Frege rejected the notion of class as primitive and as previous to the notion of function. The introduction of functions by means of classes would have led in his system to a vicious circle. In fact, it is only possible to speak about a certain class when the objects which make it up are in some sense known. If the class is finite, then in principle it is always possible to indicate all of them. In the case of infinite sets, to do this is impossible. To know the elements of an infinite set means to be able to decide about any object of the objective domain whether it belongs to this set or not. Putting it in another way, the work of infinite sets is accomplished by help of a characteristic function (the function characterizing the set). So is named that function

which relates to every element of the characterized set any fixed object (e.g. 1 or *truth*) and to every element, not belonging to the given set, any other fixed object (e.g. 0 or *falsehood*). Every characteristic function may be regarded as a logical function. Above we said that Frege was the first to introduce logical functions. Precisely this expansion of the notion of function made it impossible to define (arbitrarily) a function by means of the notion of set, because sets themselves are defined by functions. It appears very probable that Frege realized this circle and, in order to avoid it, took as primitive notion not the set, but the function. In any case, he proceeds in the 159
following way: he introduces the notion of function without any formal definition, explaining it by means of examples and hints and by this he defines the notion of range of values. In this approach Frege thought, probably, that it is improper to speak about the range of values of a function as its graphic, i.e. set of couples (nevertheless he implied that in fact it is so spoken of).* The same is true in reference to the concept and its extension. Concept (property) was introduced without definition and by it, as by the characteristic function, was defined the extension of the concept. Frege regarded such an approach as the only valid and the only one which corresponds to the real logical situation. This explains too why Frege did not use the word "*Menge*" (set, plurality, totality) preferring to it the expression "extension of a concept".

But Frege saw that the principle, by means of which the ranges of values of functions (and particularly, the extensions of concepts) are introduced, is not determined unambiguously if there exists even one function taking different values for different arguments (and differring from the identical function). In fact, let such a function be $\chi(\xi)$. Its arguments may be any object, including $\grave{\varepsilon}\phi(\varepsilon)$ and $\grave{\alpha}\psi(\alpha)$. The function $\chi(\xi)$ puts the objects $\chi(\grave{\varepsilon}\phi(\varepsilon))$ and $\chi(\grave{\alpha}\psi(\alpha))$ in correspondence to these. In virtue of the biunivocal character of the correspondence, and with the function $\chi(\xi)$ once fixed between arguments and values, it is clear that if it is true $\grave{\varepsilon}\phi(\varepsilon) = \grave{\alpha}\psi(\alpha)$ then it is true $\chi(\grave{\varepsilon}\phi(\varepsilon)) = \chi(\grave{\alpha}\psi(\alpha))$, and vice versa. Therefore it is true

$$\forall \mathfrak{a}((\phi(\mathfrak{a}) = \psi(\mathfrak{a})) \equiv \chi(\grave{\varepsilon}\phi(\varepsilon)) = \chi(\grave{\alpha}\psi(\alpha))$$

* In the Russian text the last sentence from "in this approach" is not quite clear.

which shows that for objects of the kind $\chi(\hat{\varepsilon}\phi(\varepsilon))$ the same criteria of distinction are fixed as for objects of the kind $\hat{\varepsilon}\phi(\varepsilon)$; this means that there are no means of distinguishing these objects. "Now, how is this indetermination to be eliminated?" asks Frege. "In the following way: by defining for every function, at the time of its introduction, which values it receives for classes as arguments, as well as for every other argument."[58] In fact, if this would succeed, then an object of the form $\hat{\varepsilon}\phi(\varepsilon)$ could always be distinguished from all other objects, since for every given range of values there would be a determined value for any logical function, which would mean that we dispose of an answer to any question concerning the object $\hat{\varepsilon}\phi(\varepsilon)$, in particular to the question whether it coincides or not with the object $\chi(\hat{\varepsilon}\phi(\varepsilon))$. This program could be done in the following way:
160 first by introducing some functions as elementary, having computed their algorithms[59], and afterwards by describing the rules of the construction of functions from one another. Thus, the notion of function would be fully defined. Frege, too, took this path but he was not able to follow it to the end; in the final analysis he fell into a contradiction.

Nevertheless, Frege's attempt played a role in the history of the foundations of mathematics. Therefore, the study of the proportion of constructivistic and non-constructivistic elements in his system is of great interest.

In contemporary writings, Frege is generally presented as being a Platonist. A. Church in his *Introduction to Mathematical Logic* (1956) calls Frege a "thoroughgoing Platonic".[60] This appreciation of Frege is shared by certain Soviet scholars. E. Kol'man speaks of Frege as a "follower of objective idealism".[61]

To what extent is this evaluation right? First of all let us note the following. Representatives of the subjective idealistic tendency in the philosophy of mathematics frequently accuse materialist mathematicians of Platonism. If a mathematician acknowledges the objective reality of classes, this is frequently enough for accusing him of Platonism. Soviet mathematicians have often heard from abroad the accusation of Platonism. Now Frege, as we know, was convinced of the objectivity of classes (sets) of objects. The set (class of objects) in Frege is identical to the extension of the concept,

understood as the common property acquired by means of the process of abstraction.

More. In the logical system of Frege concepts play a great role. He builds the logical calculus as *Begriffsschrift*. Frege – a categorical enemy of empiricism and nominalism – frequently declared that "a concept is logically prior to its extension."[62] He considered as a serious mistake "the superficial view as to the concept (one might call it a mechanical or quantitative view) that comes out also in Euler's diagrams".[63] In another place Frege says: "I do, in fact, maintain that the concept is logically prior to its extension; and I regard as futile the attempt to take the extension of a concept as a class, and make it rest, not on the concept, but on single things."[64] 161

Thus, does it result that, from Frege's point of view, classes have their foundation not in the individual things but in the concept? Such an assertion would completely justify accusing Frege of Platonism, if what Frege means by concept were not previously clarified. That's why we have so in detail considered Frege's theory of concept. Now we can affirm with full justification: Fregean concepts do not belong to the world of spirit, of thinking, and are something objective, existing in reality itself. Frege understood by "concept" the common property of things. He did not doubt either the objectivity of things or the objectivity of their common properties.[65]

Frege assigns objects, functions and concepts to the domain of the *designata*. In the domain of the *designators* (language) correspond to them proper names (a particular case of which are, for Frege, sentences), signs of functions and concept-words. The *senses* of proper names, of sentences, of signs of functions and names of concepts make up the content of the *proposition*. Objects, functions and concepts belong – according to Frege's expression – to *one level*: the level of objectivity. Therefore the Fregean assertion about "the priority of the concept over its extension", his statement that the class has its foundation in the concept and so on, are not idealistic positions but, on the contrary, have a rational sense. Since for Frege every class is the extension of a concept, the class is not a mechanical aggregate of individuals; the class is a set of things having a common property. Precisely in the concept (common property of things) is included the foundation of its being as a *class*. Frege

wrote that "the extension of a concept is constituted in being not by the individuals, but by the concept itself, i.e. by what is asserted of an object when it is brought under a concept."[66] In logic, insisted Frege, it is impossible to consider classes as mere mechanical aggregates of things, not united by the presence of a common pro-
162 perty. Without any doubt, Frege was right in considering that from a *logical* point of view, classes must be regarded as defined by the common properties of their elements.

Frege was an enemy of empiricism, nominalism and subjectivism. His views essentially differed from the point of view of nominalist-empiricists who deny the reality of the universal. This is his indisputable contribution to the philosophy of mathematics.

Dialectical materialism, which first in the history of philosophy gave a scientific solution to the question of the relation of universal and particular, also started, precisely, from the objectivity of the universal.

Lenin said that it is impossible to reject the objectivity of the universal in the particular and singular. He wrote: ". . . the particular does not exist otherwise than in relation to the universal. The universal exists only in the particular and through the particular. Every particular is (in one way or another) universal. Every universal is a part, aspect or essence of the particular. Every universal only approximately contains all particular objects. Every particular is incompletely found in the universal, etc., etc. Every particular is bound up with another *type* of particular (things, phenomena, processes) by thousands of transitions. Etc. *Already here* there are elements, rudiments, of the concept of *necessity*, of the objective bond up of nature, etc."[67] Dialectical materialism considers that the properties of things exist objectively.

Frege was right in acknowledging the objective character of the functional dependence, of the common properties of objects and of the relations among them. In this consisted the strength of his views. But he did not understand the dialectics of the universal and the singular, of the objects and their properties, of the things and the objective relations in which they enter.

In view of all that has been said, to the question: was Frege a Platonist? in our opinion, it is possible to give the following answer.

Under Platonism is generally meant the acknowledgment of the existence of ideal beings of a universal character, defining a world of individual things. It is evident that in Frege there is nothing of this. Therefore, one is not justified in considering Frege a representative (much less a follower!*) of this tendency. But, in Frege there was a definite exaggeration of the role of the universal, a certain separation of the common properties from the objects in which these common properties exist and reveal themselves, a lack of precision concerning the relation between object and concept. It is known that if a scholar wants to proceed according to the materialistic theory of universal properties and concepts, he must give a fully defined answer to the question about the relation between objects and universal properties, revealing the connection between universal and essence, by the law of evolution of objects. At the same time, for Frege, in this reference there is much vagueness. In spite of all his explanations, the reader of his works cannot get rid of a strange feeling: it always seems as if the author had not finished saying something; thus one would like to ask what is, essentially, the Fregean concept. From Frege's explanation it follows that it is, in a sense, the common property of things; but Frege does not say this openly. Furthermore, why does Frege take the term *concept* in a sense different from the usual (concept = the mental reflection of things)? Why does he so vaguely answer the question about the relation between objects and common properties? 163

This obscurity, this incompleteness in the theory of concept of Frege, and also the peculiarity of his terminology are, apparently, in fact a concession to the tendency of philosophy which has the name of "moderate realism". It could not have been otherwise in a thinker not understanding dialectics. As we show below, Frege's views had a metaphysical character.

Nevertheless it is possible to speak of certain *materialistic* tendencies of Frege's logic. These tendencies appear not only in Frege's unrelenting struggle against scepticism, subjective idealism and solipsism, in the struggle against formalism and "creative definitions" in mathematics, but also in his direct acknowledgment of the

* The expression between parentheses is not quite clear in the Russian text.

objectivity of the external world. Frege did not doubt that outside of human consciousness there is a real world, that the objects of this world are conceived by means of the organs of sensation and thought. But the spontaneously materialist views of Frege were not consequent. In a series of points (e.g. in the question about the origin of the arithmetical truths) the inconsequence of Frege, his ignorance of dialectics, brought him to idealism.

The strong and weak sides of Frege's philosophy of mathematics clearly appeared in the Fregean notion of *truth*. Frege was a categorical defender of the *objectivity of truth*. He constantly repeated that truth is objective, that it does not depend on the man who states the judgment. Therefore, he categorically came out against the subjectivist-psychologistic theory of judgment as a connection of ideas (*Vorstellungen*). For him – a man considering judgment not as an empty shell of thought but as the acknowledgment of its objective truth – the subjective approach to truth was completely inadmissible. *To be true* [*Wahr sein*] and to consider as true [*Für wahr halten*], Frege stressed, these are different things.

But we know that truth and falsehood were considered by Frege as particular objects (truth-values). Frege thought that among objects some are sensibly perceptible things, others are what Frege calls *logical objects*. "We can distinguish physical objects from logical ones, whereby, obviously, no exhaustive division is intended. The former are real in the genuine meaning of the word; the latter are not, but, nonetheless objective; it is true that they cannot affect our senses but must be comprehended through our logical abilities. Our numbers are such logical objects."[68] For example, the
164 number *one* is something identical for everybody, appearing to all in an identical way. It does not at all depend on who expresses a judgment about it. In the group of logical objects Frege also introduced the extensions of concepts (sets of objects). Truth and falsehood: these are also objects. Therefore, for Frege, truth cannot be a part of a proposition, since it belongs to the domain of the objective. He writes: "A truth-value cannot be a part of a proposition anymore than, say, the Sun can, for it is not a sense but an object."[69]

Frege's attitude on logical objects, in the number of which he introduced falsehood and truth, contains a rational nucleus, which

is the acknowledgment of the objective value of scientific abstractions. From the logical point of view such an extended conception of objects is completely justified and it is widely used in contemporary logic. Thus, in the construction of truth-tables of classical sentential calculi, truth and falsehood are considered as objects, and the operations of the calculus (conjunction, disjunction, implication, etc.) as functions relating truth and falsehood to truth and falsehood. Many contemporary authors in the construction of logical calculi start from the admission of a special objective domain, consisting of two "selected" objects: truth and falsehood. This means that truth and falsehood in the corresponding logical system are considered as given, so to speak, as "ready made". Such an approach is admissible and gives positive results in logic.

But Frege did not see that the treatment of truth and falsehood as objects is derived from the notion of truth as correspondence of our knowledge with objective reality; he did not understand that the question about truth, about the adequate and faithful reflection of the material world in human consciousness is solved, in the final analysis, not by the formation of logical systems (however great their role might be in knowledge) but by practice, because the criteria of practice are in the end decisive in the most abstract domains of science.

IV. THE CONTRADICTORY CHARACTER OF FREGE'S SYSTEM

Above we said that in his logic Frege acknowledged the existence of a universal objective domain, including all the objects of the world. This view of Frege originated in his general philosophical conception, precisely in his understanding of logical laws. 165

Frege started from the right idea that every law, speaking about *what there is* in reality, can be considered as a rule of thinking in correspondence to this law. In this sense all laws – laws of geometry or physics no less than laws of logic – are "laws of thought".

What now is the specific character of the laws which are studied in logic? Frege gave to this question an inexact answer. He did not see the essence of the laws of logic in the fact that they are laws of the

reflection of the material world in the consciousness of man, but in the fact that they would have a universal and absolute character. "These deserve the name 'laws of thought'" says Frege "only then when it can be said that they are the most universal of all, which prescribe how thinking is to proceed wherever there is any thinking." [70]

The metaphysical character of such ideas is evident. It is true that Frege understood that "the historical approach, with its aim of detecting how things begin and of arriving from these origins at a knowledge of their nature, is certainly perfectly legitimate." [71] But for Frege it is limited. Laws of thought do not have a historical character. They have an unconditioned eternal force. They are identical everywhere and always. From Frege's point of view every law of thought prescribes "how to judge, irrelevant of the by whom, where and when of the judging." [72] If at any time thinking beings different from man were discovered, they too would think according to these laws.

Naturally, the question arises: why, by what right, do we acknowledge any logical law as true? Frege believes that logic is powerless to give an answer. Logic can only connect a given law with the other logical laws. Where this is not possible, logic leaves the question unanswered. "Stepping outside of logic", writes Frege, "one can say: we are forced by our nature and external circumstances to judge; and when we judge we cannot spurn these laws – of identity, for example; we must recognize it if we do not want to bring our thought to confusion and, finally, to renounce any judgment." [73]

Thus Frege did not understand that laws and forms of logical thought are an analogon of reality, that they reflect the most common relations of things, that they change in the course of knowledge and social-historical practice, that precisely practice is the criterion of their truth. Here appears the idealistic and metaphysical side of
166 his views.

From this Fregean understanding of the laws of logic it naturally follows that in logic one should always start from a universal objective domain among the objects of which are every possible set of objects, set of sets, set of sets of sets. In this regard the dispute between Frege and Schröder is interesting.

In his *Algebra der Logik* Schröder rejected the notion of a uni-

versal objective domain because the admission of it plus his proposed definition of "identical zero" (the empty class) would introduce into his logical calculus a contradiction. In the objective domain ("manifold") in which the given reasoning is made, Schröder imposed the following restriction: among the objects of a manifold must not be objects such that they are classes of objects of the same domain. If the initial objective domain (manifold) is designated by "M", then the classes of objects of this domain will belong to another objective domain M_1, the classes of classes of objects will constitute a new domain M_2, etc. Thus Schröder presented in his own way a theory of types. In correspondence to this theory Schröder thought that every argumentation is always done in a defined domain of objects, which should be previously shown and which cannot be universal.

Against the Schröderian restriction of the objective domain Frege reacted strongly (in the article 'Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra der Logik'). He showed that the Schröderian paradox may be escaped in another way: by means of the distinction between the relations of inclusion of a class in another class and the belonging of an object to a class of objects. Thus, the universal objective domain may be maintained.

But to Frege it seemed insufficient to show that, apart from Schröder's way, another is possible in which the admission of the universality of the objective domain is maintained. He tried to show that Schröder's solution was completely unsound, that the restriction on the objective domain proposed by the latter can in no way be maintained in his calculus. Frege was pushed to this by his persuasion that "logic has the right of claiming the unrestrictedness of the domain of application of its laws" and that, therefore, we must not in each given case in any way limit the field of their logical application.

Trying to give a foundation to his thesis, Frege came to a direct alteration of Schröder's thought (taking advantage of some inaccuracies in his exposition). Undoubtedly, Frege did not prove what had to be proved. The theory of types of Schröder, by means of which Schröder eliminated the paradox related to the identical zero in his logical calculus, was completely right; it was even, in a certain

167 sense, *better* than Frege's theory, because it excluded, too, the Russellian antinomy.[74]

Frege's introduction of an all-embracing objective domain is intimately connected with his requirement that every concept be, as he says, sharply determined, have precise limits. Frege derives such a requirement for concepts from the amplification of the circle of arguments and values of functions, and connects it to the principle according to which there should not be in science any non-signifying, designation-less expressions.

Frege considers the following example. If we speak about the arithmetic of whole numbers, a and b in $a + b$ denote whole numbers. But, what does $\odot + 1$ mean, if $\odot$ denotes the Sun? Would it be possible to consider this expression as not having reference? No, since it is a rule of scientific rigour that in science there must not be expressions lacking reference. This means that for the expression $\odot + 1$ some reference must be established. For concepts, the indicated rule means the requirement that for every argument they obtain, as value, truth or falsehood; "that it shall be determinate, for any object, whether it falls under the concept or not. In other words; as regards concepts we have the requirement of sharp delimitation; if this were not satisfied it would be impossible to set forth logical laws about them."[75] "The requirement of the sharp delimitation of concepts thus carries along with it this requirement for functions in general that they must have a value for every argument."[76] Thus, if the concept $\xi + 1 = 10$ is sharply defined, then the function $\xi + 1$ must be sharply defined too, because, if it is not sharply defined, i.e. if there is an argument for which it has no values, then the concept in question would not relate either truth or falsehood to this argument. Thus, all functions and concepts must be defined in the whole universal domain of objects, including truth and falsehood as particular objects. Specific mathematical functions must be, correspondingly, defined in such a way that the domain of their arguments be the whole objective world (Frege also proceeds in this way in his mathematical-logical system).

168 The Fregean requirement of sharp delimitation of concepts is the requirement that every concept have a constant and unchangeable

extension. In symbolic form, it takes the shape of the principle about which we have already spoken above:

$$\forall a(\phi(a) = \psi(a)) \text{ equiv } \grave{\varepsilon}\phi(\varepsilon) = \grave{\alpha}\psi(\alpha)$$

and may be considered as the assertion that every concept has an extension in the sense of an unchangeable, sharply fixed object. This principle is usually called the principle of extensionality. The acknowledgment of this principle gives Fregean logic an extensional and set-theoretical character. Although Frege did not identify the concept with its extension nevertheless, in virtue of the principle of extensionality, in his logical system concepts are considered as given if their extension is defined, i.e. if the class of objects falling under the given concept is shown. According to Frege, the logic of current contentful thinking also agrees with this principle.

In creating his mathematical-logical system, Frege aimed at founding arithmetic by means of logic. Ending the second volume of his *Grundgesetze der Arithmetik*, Frege imagined that the task which he had set for himself, in undertaking this work, was completely solved. But, at this time, Russell discovered in his system an antinomy, connected with the notion of a class of all classes and with the universal character of the objective domain supposed by Frege.

The antinomy of Russell is obtained in the following way. Frege considers the extensions of concepts (properties) as particular kinds of objects. Since the objective domain is universal, the extensions of concepts, which are objects too, enter in it along with other objects. Thus, in the universal domain of objects there are objects, sets of objects (as particular objects), sets of sets of objects (as particular objects, too), etc.

Let us introduce the notion of normal and non-normal classes (extensions of concepts, sets). We shall consider a class as normal if it does not have the property which defines it as a class. This may be written by means of the formula

$$N(\grave{\varepsilon}\phi(\varepsilon)) = \overline{\phi(\grave{\varepsilon}\phi(\varepsilon))}, \tag{1}$$

where the line above the expression means negation, $N(\ldots)$ means the property of being normal, and the letter ϕ is a variable for properties (in its place may be put their names); the expression $\grave{\varepsilon}\phi(\varepsilon)$ is

a variable term: every time that in place of the variable ϕ the name of any property is introduced the expression $\hat{\varepsilon}\phi(\varepsilon)$ becomes the name of a defined class (precisely of the class whose objects possess the property the name of which is introduced in the place of ϕ).
169 From (1) – thanks to the fact that a double negation of any expression is equivalent to the expression itself – it follows that a non-normal class is a class which, if considered as a special object, precisely has the property which defines its being as a class, i.e.

$$\overline{N(\hat{\varepsilon}\phi(\varepsilon))} = \phi(\hat{\varepsilon}\phi(\varepsilon)) \qquad (1')$$

Now let us pose the question: if we make the class of all normal classes (i.e. the class of objects which possess the property of being a normal class), then will this class have the property of being normal (i.e. will it be a normal class) or not? In order that this question have a meaningful character, we need a series of assumptions. First of all, we must admit that the property *to be normal* has an extension, i.e. that the class of all normal classes does exist. This assumption is accepted by Frege's theory, since all concepts in it have an extension. Moreover, it is necessary to admit that the class of all normal classes itself is an object, since otherwise any question about its properties would be meaningless, e.g.: is it normal or not (by this we assign to an object that it has or may have properties, and is or may be in certain relations to another object). This assumption is accepted also in Frege's theory, since in it the extension of every concept is a special logical object. Finally, we need the assumption that the class of all normal classes is of the same type as the other objects of the objective domain – of the same type in the sense that, for it, as for other objects of the objective domain, it is meaningful to put the question whether it has the property of being normal or not. This property, too, is presupposed in the theory of Frege, because in it all objects constitute one universal domain and this means that all properties are considered to be in this domain. Therefore, the question about the property *to be normal* can be posed in regard to any object whatsoever, also in regard to the class of all normal classes.[77]

Which is now the answer in Frege's theory to the question asked? The answer happens to be contradictory. Let us put in the formula

(1) in place of the variable ϕ the name of the property *to be a normal class* i.e. N. We obtain:

$$N(\hat{\varepsilon}N(\varepsilon)) = \overline{N(\hat{\varepsilon}N(\varepsilon))} \qquad (2)$$

This formula from Frege's point of view is a meaningful expression having a reference because it is a well-formed formula. As a matter of 170
fact, it is built according to the rules of his calculus, and in it there are no meaningless expressions. The expression $\hat{\varepsilon}N(\varepsilon)$ is meaningful and designates the class of all normal classes, the existence of which is not excluded in Frege's theory. The expressions $N(\hat{\varepsilon}N(\varepsilon))$ and $\overline{N(\hat{\varepsilon}N(\varepsilon))}$ are meaningful, too, because in Frege's theory the class of all normal classes is an object of the universal domain, which means that just as a sentence about the inherence of any property in this object, e.g. the property N, i.e. the sentence $N(\hat{\varepsilon}N(\varepsilon))$, so too the negation of such inherence, i.e. the sentence $\overline{N(\hat{\varepsilon}N(\varepsilon))}$ necessarily is meaningful and is either true or false. In virtue of this, every expression (2) has a meaning and must be either true or false.

What is the meaning of this formula?

The expressions $N(\hat{\varepsilon}N(\varepsilon))$ and $\overline{N(\hat{\varepsilon}N(\varepsilon))}$ are sentences. For Frege's theory sentences are names of a special kind of objects, the truth-values. The equality sign, for Frege, means a full coincidence (identity) of the objects, the names of which are to the left and right of it. Whence the content of the formula (2) consists in the assertion that the sentences $N(\hat{\varepsilon}N(\varepsilon))$ and $N(\hat{\varepsilon}N(\varepsilon))$ are either both true or both false; this means that the content of the sign "=" in the given case coincides with the meaning of the equivalence of sentences in the propositional calculus, which may be regarded as a double implication (implication in both directions). Therefore, formula (2) may be rewritten as follows:

$$(N(\hat{\varepsilon}N(\varepsilon)) \to \overline{N(\hat{\varepsilon}N(\varepsilon))}) \;\&\; (\overline{N(\hat{\varepsilon}N(\varepsilon))} \to N(\hat{\varepsilon}N(\varepsilon))) \qquad (3)$$

where "$\to$" denotes implication (the logical connective "if . . . then") and " &" conjunction (the logical connective "and"). Formula (3) may be read: if a class corresponding to the property of being normal is itself normal, then it is not normal; if it is not normal, then it is normal.

We have a contradiction: the class of all normal classes is normal

and is not normal. Now, let us symbolize the sentence $N(\hat{\varepsilon}N(\varepsilon))$ by the letter A. Then formula (3) takes the form $(A \to \overline{A})$ & $(\overline{A} \to A)$. Because $\mathfrak{A} \to \mathfrak{B}$ is equivalent to $\overline{\mathfrak{A}} \vee \mathfrak{B}$, where "$\vee$" is the disjunction sign (the logical connective "or" in the non-exclusive sense), we obtain: $(\overline{A} \vee \overline{A})$ & $(\overline{\overline{A}} \vee A)$. Since $\overline{A} \vee \overline{A}$ is equivalent to $\overline{A}$, and $\overline{\overline{A}}$ equivalent to A, we have finally:

$$\overline{A} \ \& \ A \tag{4}$$

i.e. a contradiction. So is revealed the sense of formula (2).

But Formula (2) and also then formulae (3) and (4) in Frege's system must be acknowledged as true, because they follow from
171 formula (1), which is true by definition, by means of the transformations which produce true from true. Thus, a contradictory statement is deducible in Frege's system. Such is the antinomy – a contradiction ("paradox") not eliminated from the system (without changing the system) – which Russell discovered in Frege's calculus.

Having discovered a contradiction in Frege's system, Russell spoke of it in a letter to Frege. Russell's letter made a very strong impression on Frege. In the epilogue to the second volume of *Grundgesetze der Arithmetik*, especially written on this antinomy, Frege says: "Hardly anything more unfortunate can befall a scientific writer than to have one of the foundations of his edifice shaken after the work is finished. This was the position I was placed in by a letter of Mr. Bertrand Russell, just when the printing of this volume was nearing its completion." [78]

Frege saw the cause of his failure in the fact that he had operated on the assumption that for every concept there is an extension which may be isolated by means of the "principle of abstraction". Thereby he conceived extension as a constant, strictly fixed object, not containing any indetermination. "And even now I do not see how arithmetic can be scientifically established; how numbers can be apprehended as logical objects, and brought under review, unless we are permitted – at least conditionally – to pass from a concept to its extension. May I always speak of the extension of a concept – speak of a class? And if not, how are the exceptional cases recognized? Can we always infer from one concept's coinciding in extension with another concept that any object that falls under the one

likewise falls under the other? These are the questions raised by Mr. Russell's communication . . . What is in question is not just my particular way of establishing arithmetic but whether arithmetic can possibly be given a logical foundation at all.[79]

The discovering of the antinomy by Russell produced such a strong effect on Frege that, using a metaphor of D. Hilbert, he left the battle-field. After publishing the second volume of *Grundgesetze der Arithmetik*, Frege essentially did not come back any more to the questions of foundations of arithmetic.

Hilbert gave the following account of the situation in which Frege found himself: "Gottlob Frege proposed to establish the laws of arithmetic by means of logic taken in its traditional sense . . . But, 172
in remaining true to his basic thesis that a concept (set) is defined and immediately applicable if and only if for every object it is decided whether or not it falls under the concept, and in not therein restricting the use of "every" in any way, he lays himself open to paradoxes of the set-theory such as are contained in the concept of class of all classes."[80]

Frege did not see a way out of this situation. This is explained by Frege's "naive" set-theoretical attitude, together with the metaphysical opinion that "logic has the right of claiming an unrestricted domain of application for its laws". Frege's tendency to the introduction in logic of a universal domain of objects, which would embrace all the objects of the world, was wrong. Further, Frege carried too far the idea on the absolutely universal character of the laws of logic, on the all-embracing character of the domain with which logic would always have to do, on the unchangeability and constancy of its objects which, for him, should not include any indeterminacy or vagueness, on the inapplicability of the categories of change to the logical-mathematical systems. There was the imprint of metaphysical narrowness on Frege's views. Frege also did not understand the dialectic of the evolution of the basic concepts of mathematics and logic. Therefore, Russell's discovery of this paradox brought Frege to the conclusion that his system had failed.

This conclusion certainly did not have any basis: Frege did not understand that his system admitted of a further evolution. The history of mathematical logic after Frege showed that the over-

coming of the Russellian kind of antinomy is possible only by means of a renunciation of the universal objective domain and the introduction of movement into logical systems. The ultimate cause of the failure of Frege is rooted precisely in the lack of understanding of movement in logic.

V. FREGE'S CONTRIBUTION IN THE FOUNDATIONS OF MATHEMATICS AND MATHEMATICAL LOGIC

In the series of remarkable people of the nineteenth century who created mathematical logic (Boole, Schröder, Poreckij, Peano and others) Frege has a prominent place. His greatest contribution is the creation of the first formal logical system in the history of science and its application to the foundations of arithmetic.

Frege formulates in the following way the properties of a formal mathematical-logical system. For the construction of a formalized arithmetic "in the execution of a proof considerably higher demands
173 for rigor must be made than is usual in arithmetic. A set of a few methods of deduction has to be fixed beforehand, and no step may be taken which is not in accordance with them. Consequently, when passing over to a new judgment we must not be satisfied, as mathematicians seem nearly always to have been hitherto, with saying that the new judgment is evidently correct, but we must analyse this transition into the simple logical steps of which it is composed – and often there are not a few of these simple steps. No hypothesis can thus remain unnoticed. Every axiom which is needed must be discovered, and it is just the hypotheses which are made tacitly and without clear consciousness that hinder our insight into the epistemological nature of a law.

In order that such an undertaking should succeed the concepts which we need must naturally be conceived distinctly."[81]

The constructions of Boole and Schröder were not formal systems in the contemporary sense, although they were also built as calculi. They were algebrae with a contentful character. Using the contemporary terminology, it may be said that they were not formalized languages. Frege built the first *formalized* language and, although his system happened to be contradictory, this does

not diminish his merit as a man opening a new path in science.

Frege's contribution to mathematical logic and the foundations of mathematics has many aspects. He explained the concept of variable and created an original theory of functions. He widened the notion of function, giving up the idea that only numbers can be arguments and values of functions; he elaborated (independently from Peirce) the notion of truth-value. Frege possesses the merit of having introduced in mathematical logic the notion of logical function and the idea of properties and relations as special cases of functions. He was the first to introduce in a clear way quantifiers in mathematical logic and to start using them systematically. He introduced the concepts of first, second and so on types and showed the logical legitimacy of "empty concepts". Frege's reflection on the concept of existence in logic is, without any doubt, of interest. Frege clearly formulated the principle of abstraction, and explained the significance of the complete mathematical induction. To him is due the earliest deductive axiomatic system of sentential calculi (in the work of 1879). Frege proposed an original definition of the concept of number. His formal arithmetic is historically the first formalization of a concrete mathematical theory. Frege's works in foundations of arithmetic opened the way to further research in this domain.

Frege was the first to show the great significance for logic and 174
mathematics of the study of the relation between name and object – the name relation – and made the first attempts to build a theory of the content of proper names. He was the founder of the theory of descriptions and the first to introduce in the logical calculus the operator "that, which" used for the formation of names of objects from concept-words (names of concepts).

Frege's reflection on the nature of the sign and its role in knowledge is very valuable from the gnoseological point of view. He categorically supported the view that the objects of mathematics and logic are not at all signs (as the formalists tried to show) but signs are intended only to help the mutual understanding of men, and man's understanding with himself, helping memory. Therefore, for every sign, name and description it is necessary to demand that they designate something. Frege constantly stressed that names not designating any object are not allowed in science.

The greatest merit of Frege is actually the distinction, contained in his work, between formal language and metalanguage by means of which is accomplished the construction of a given formal language (of course Frege did not use these terms). He was the first in mathematical logic to clearly distinguish sentences of the logical calculus (and, in part, axioms) from the rules of this calculus, formulated in the informal language.

In Frege's works many questions are raised which have not up to now been solved. Thus, the introduction by Frege of the notion of the sense of linguistic expressions, brought up a whole complex of problems of contemporary logical semantics. The notion of sense occupies an important place in the semantic researches of recent years. The problem of the expression of sense by means of formal systems acquired a particular significance in relation to the evolution of mathematical linguistics, in particular in connection with the creation of linguistic intermediaries in automatic translation from one language to another.

Frege was a violent opponent of subjectivism and formalism. His criticisms of these tendencies have not lost their value even today. Frege's struggle against "creative definitions" in mathematics had a historically progressive character. Frege's defense of the objectivity of sets, his categorical stress on the objectivity of truth – along with the acknowledgment of the objective character of the external world – are evidence of the materialistic tendencies of his logical views, and of his philosophical approach to mathematics. In prin-
175 ciple Frege took a basically correct position in the question about the correspondence between ordinary and formalized languages. He thought that the logical calculus (*Begriffsschrift* in Frege's terminology) is only an auxiliary method conceived for certain scientific purposes and not substituting at all for either contentful thought or living human language.

Frege dealt with philosophy in a way different from contemporary positivists, who actually reject philosophy. In the preface to the book *Grundlagen der Arithmetik*, devoted to the investigation of the concept of number, Frege wrote that "any thorough investigation of the concept of number, is bound always to turn out rather philosophical. It is a task which is common to mathematics and

philosophy."[82] He did not belong to those mathematicians who, in his own words, "do not like to enter in the dangerous ways of philosophy". In the *Grundlagen der Arithmetik*, not only the mathematical theories of number are considered, but the philosophical and perception-theoretical ones too. Frege wrote: "I permit myself the hope that even the philosophers, if they examine what I have written without prejudice, will find in it something of use to them."[83] Frege as philosopher was head and shoulders above his logistical successors.

This high appreciation of Frege's merits does not mean that he had no failures and errors. It could not be otherwise for a thinker in a wholly metaphysical position and admitting idealistic hesitations. One of these mistakes was the idea of the possibility of a pure logical foundation of mathematics. Frege thought that arithmetic is a further developped logic, that the rigorous foundation of arithmetical laws reduces itself to purely logical laws and to them alone. Therefore, the arithmetical language may and must be built as a logical one. In the framework of his mathematical-logical theory he wanted to develop "as from one embryo" from eight primitive functions, having a logical character, all the richness of objects and functions dealt with in arithmetic.[84] In contemporary mathematical logic he was the real founder of the theory of the reduction of mathematics to logic, the founder of the tendency in the foundations of mathematics which afterwards received the name of logicism.

In the article 'Mathematik als Zweig der Logik', written in 1930, 176
Rudolf Carnap (supporting then the point of view of logicism) explained in the following way the essence of this tendency: "The basic idea of logicism may be formulated as follows: mathematics is a branch of logic. This means that there are no specifically mathematical, extra-logical primitive concepts and laws. The concepts of mathematics can be derived from logical concepts, i.e. from the concepts which are necessary for the construction of ordinary, non-mathematical logic. The propositions of mathematics form a sector of logical propositions.

This basic idea, which is related to Leibniz's views, was clearly formulated for the first time by Gottlob Frege. He also provided the derivation of some of the primitive concepts of mathematics from those of logic."[85]

We know that the efforts of Frege ended in defeat. In our time the failure of the reduction of mathematics to logic may be considered as proved.[86] In this sense Frege was mistaken. There is no doubt as to the erroneousness of the methodological aspect of this conception of Frege; he wrote: "arithmetic is a branch of logic and need not borrow any ground of proof whatever from experience or intuition."[87]

But, at the same time, another thing should be mentioned. Frege's views are different from the logicism of Russell and his followers, because from Frege's point of view logical concepts – by means of which, for him, are derived all the concepts of mathematics – have an objective foundation. Logic, and together with it mathematics, from Frege's point of view, has a direct relation to the real world. Frege always stressed that mathematics must be applied to reality. Mathematics for Frege is not similar to mathematics for Russell, for whom it is something in which one knows neither that about which it speaks, nor when that which it says is true.

The limitations of Frege's points of view were clearly seen in the discussion between him and Hilbert about the question of the foundations of geometry. They hardly understood each other. Frege did not agree with the philosophical features of formalism and in this he was surely right. But, at the same time, he did not understand that the work of Hilbert on the foundations of geometry had another and basic aspect, that in it geometry was settled on a rigorous, scientific basis.

177 The weak aspects of Frege's views were exploited by a series of idealist philosophers (Husserl's phenomenology). Idealism also used some of the scientific discoveries of Frege, adapting them to its needs (Husserl's use of the Fregean notion of *sense*).

The eminent place occupied by Frege in logic and mathematics was not understood by his contemporaries. It was necessary that a decade pass before the richness of ideas contained in his works began to be completely clear. With the evolution of mathematical logic, the historical importance of Frege became more and more clear.

The following comparison will show how the appreciation of Frege has changed in the course of the last decades.

In the thick volume published in 1920 by Th. Ziehen, *Lehrbuch der Logik*[88], Frege is very superficially mentioned. In the fundamental passage, the author devotes to him less than half a page. Ziehen gives no importance to the works of Frege. Such an appreciation of Frege by the author of the German *Lehrbuch* – typical representative of the idealistic formal logic – is perfectly understandable. But mathematicians already in these years had another appreciation of Frege. In the year 1925, reporting in a mathematical congress, organized in memory of Weierstrass, David Hilbert said that Frege had done very much for the foundations of mathematics. In the work *Problems of the Foundations of Mathematics* (1923) Hilbert called Frege a great classic and one of the creators of research on the foundations, putting him on the same level with Cantor and Dedekind. In 1931 the German logician and historian of logic, H. Scholz, wrote that Frege, without any doubt, was the greatest logician of the nineteenth century.[89]

In our time the importance of the scientific contribution of the remarkable German mathematician and logician has begun to be completely evident. Contemporary works in mathematical logic contain a great number of quotations from Frege's works. His works are being reprinted. In the annotation to a collection of selected works from Frege about questions of logic and philosophy of mathematics, edited in Oxford in 1952, it is said that nobody since Aristotle did so much for the evolution of formal logic as Frege did.[90] If such an evaluation appears exaggerated, it cannot be considered a great mistake; it is rather an act of justice to a man whose contribution to science for a long time was underestimated.

REFERENCES

1. The author expresses his great gratitude to Sofija Aleksandrov Janovskaja for the scientific guidance and help given in the preparation of this article.
2. H. Poincaré: 'Du rôle de l'intuition et de la logique en mathématique'. *Comptes Rendus du IIème Congrès Internationale des Mathématiciens, 1900*, Paris, 1902, pp. 120–122.
3. In addition to this, Frege wrote the following articles: 'Über Anwendungen der Begriffsschrift' (1879), 'Über die wissenschaftliche Berechtigung einer Begriffsschrift' (1882), 'Über den Zweck der Begriffsschrift'

(1882), 'Über formalen Theorien der Arithmetik' (1886), 'Über die Begriffsschrift des Herrn Peano und meine eigene' (1897), 'Der Gedanke' (1919), 'Die Verneinung' (1919), and others. Of great interest are the letters of Frege to the German mathematician Liebmann, on the foundations of geometry, the answer of Hilbert and another letter of Frege (1900).

4. Concerning Dedekind, Frege was wrong. See *The Journal of Symbolic Logic* 22 (1957) 145–158.
5. G. Frege: *Grundgesetze der Arithmetik. Erster Band.* Jena, 1893, S. XI (in following notes this will be designated *Grundgesetze I* with indication of pages).
6. See the collective work *The Philosophy of Bertrand Russell*, Chicago-London, 1944.
7. Having begun his philosophical evolution with objective idealism, Russell passed over to subjective idealism.
8. *Grundgesetze I*, pp. XIX–XX.
9. *Ibid.*, p. XXII.
10. V. I. Lenin: *Works*, Vol. 14, p. 44 [*Materialism and Empirio-Criticism*, Foreign Languages Publishing House, Moscow, 1952, p. 49, chapter 1, section 2].
11. *Grundgesetze I*, p. XXIII.
12. *Grundgesetze I*, p. XXIV (my italics, – B. B.).
13. G. Frege: *The Foundations of Arithmetic. A logico-mathematical enquiry into the concept of number.* Oxford, 1950, p. X (In the following notes this is designated: *Frege 1950*, plus page indication).
14. *Ibid.*, p. VII.
15. *Grundgesetze I*, p. XIX.
16. *Grundgesetze I*, p. XXI.
17. G. Frege: 'Über Sinn und Bedeutung', *Zeitschrift für Philosophie und philosophische Kritik* 100 (1892) 31 (In the following notes this is designated: *Frege 1892*, plus page indication).
18. *Grundgesetze I*, p. XVII.
19. *Frege 1892*, p. 32, note.
20. *Grundgesetze I*, p. XIII.
21. *Frege 1950*, pp. 107–108.
22. *Grundgesetze I*, pp. XIII–XIV.
23. G. Frege: 'Kritische Beleuchtung einiger Punkte in E. Schröder's *Vorlesungen über die Algebra der Logik*', *Archiv für Systematische Philosophie* I (1895) 456 (In following notes this is designated: *Frege 1895*, with page indication).
24. *Frege 1895*, p. 449.
25. Unbekannte Briefe Freges über die Grundlagen der Geometrie und Antwortbrief Hilbert's an Frege. Aus dem Nachlaß von Heinrich Liebmann herausgegeben von Max Steck. *Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Mathematisch-naturwissenschaftliche Klasse. Jahrgang 1941, 2. Abhandlung.* Heidelberg, 1941, p. 12 (In following notes this is designated: *Briefe*, plus page indication).
26. G. Frege: *Grundgesetze der Arithmetik. Zweiter Band.* Jena, 1903, p. 154. (In following notes this is designated: *Grundgesetze II*).

27. *Ibidem.*
28. *Grundgesetze I*, p. XIII.
29. *Grundgesetze II*, p. 101.
30. *Ibidem.*
31. *Grundgesetze I*, p. 2 (italics mine – B. B.)
32. *Frege 1895*, p. 452 (italics mine – B. B.)
33. An object, for Frege, "falls under a concept", if, having asserted this concept about the object, we obtain truth. See below.
34. *Frege 1895*, p. 452.
35. Grundgesetze I, p. IX (italics mine – B. B.)
36. G. Frege: 'Über Begriff und Gegenstand', *Vierteljahrschrift für wissenschaftliche Philosophie* 16 (1892) 201–202. (In the following notes this is designated: *Frege 1892*, plus indication of page)
37. *Briefe*, p. 9.
38. *Frege 1892*, p. 193.
39. G. Frege: *Funktion und Begriff*, Jena, 1891, p. 2 (In following notes this is designated: *Frege 1891*, with page indication).
40. It is interesting to note that Russell regressed from Frege and identified again the function with its analytical expression.
41. *Frege 1891*, p. 1. [This reference is wrong; not having found the corresponding original text, we directly translate what the author gives as quotation].
42. G. Frege: 'Was ist eine Funktion?' In: *Festschrift Ludwig Boltzman gewidmet zum 60. Geburtstage, 20. Februar 1904*, Leipzig, 1904, p. 658 (In following notes this is designated: *Frege 1904* plus page indication).
43. *Frege 1904*, p. 665.
44. *Grundgesetze I*, p. 7.
45. For the explanation of this see below.
46. G. Frege: *Funktion und Begriff*, Jena, 1891.
47. In another place [*Grundgesetze I*, § 9, p. 10] Frege observes that the Leibnizian-Boolean calculus was, as a whole, based on this law.
48. *Grundgesetze I*, p. 7.
49. *Grundgesetze I*, p. 8. Frege is speaking about a function with one argument.
50. *Frege 1891*, p. 16.
51. The assumption of the universality of the objective domain is only a sufficient (not necessary) condition of such an interpretation. See below.
52. This does not mean that Frege was aware of the true philosophical solution of the question about the universal and particular. About this, we shall speak later.
53. For the sake of simplicity, we limit ourselves to the consideration of whole numbers.
54. If classes are considered in different objective domains, then the Fregean "extension of the concept" does not coincide with the informal or contentful (*soderžatel'noe*) notion of class. Thus if we take the concept $\xi > 2$ in the domain of natural numbers and in the domain of integers, then the extension of this concept, considered in one domain, is not the same, according to Frege's definition, as the extension of the same concept considered in the other domain; but, if the extension of a concept

is understood as a class of objects, then the extension of the given concept happens to be the same in both domains.

55. *Briefe*, p. 9.
56. *Frege 1892*, p. 204.
57. Before Dirichlet, Lobačevskij had such a conception of the function.
58. *Grundgesetze I*, p. 16.
59. This program must not necessarily be strictly constructivistic. Given the function, it is possible to suppose some problems solved, i.e. to use not absolute algorithms but the so-called algorithm of reducibility.
60. A. Church: *Introduction to Mathematical Logic*, Princeton, 1956. § 04, note 66.
61. E. Kol'man: *Predmet i metod sovremennoj matematiki* [Object and Method of Contemporary Mathematics], Moscow, 1936, p. 40.
62. *Frege 1895*, p. 455.
63. *Ibidem*, p. 441.
64. *Frege 1895*, p. 455.
65. That, for Frege, properties were precisely properties of objects, is clear, for example, from his views on the ways of proving the consistency of mathematical statements. In the letter to Hilbert Frege says: "what means do we have to prove that certain properties or conditions (or whatever one calls it) are not in contradiction with one another? The only one I know is this: to indicate an object which possesses all these properties; to give a case where these conditions are completely fulfilled. Non-contradictoriness does not seem to be provable in another way."
66. G. Frege: Kritische Beleuchtung einiger Punkte in E. Schröder's Vorlesungen über die Algebra der Logik. In: *Archiv für systematische Philosophie* 1 (1895), p. 451. From Frege's point of view we assert in a sentence some property (designated by the grammatical predicate: the concept-word) about an object (designated by the grammatical subject).
67. V. I. Lenin: *Filosofskie Tetradi* [Philosophical Notebooks], 1947, p. 329.
68. Frege, *Grundgesetze II*, p. 86.
69. *Frege 1892*, p. 35.
70. *Grundgesetze I*, p. XV.
71. *Frege 1950*, p. VII.
72. *Grundgesetze I*, p. XVII.
73. *Ibidem.*
74. About the antinomy of Russell, see below.
75. *Frege 1891*, p. 20.
76. *Ibidem.*
77. We are not considering here the other assumptions required for obtaining the Russellian antinomy. To them belong, e.g. the authorization to write definitions (the principle of reduction). We leave this aside, because in our problem does not enter either the consideration of all the conditions of obtaining this antinomy or a *resumé* of the different solutions of it.
78. *Grundgesetze II*, p. 253.
79. *Ibidem.*
80. D. Hilbert: *Die Grundlagen der Geometrie* [References are given to the Moscow–Leningrad edition of 1948, pp. 323–324].

81. *Grundgesetze I*, p. 1.
82. *Frege 1950*, p. V.
83. *Frege 1950*, p. X.
84. Even if Frege's attempt had been successful, this would not have meant that he would have succeeded in providing, by means of his system, for the whole of contentful arithmetic. In the year 1931, K. Gödel showed the impossibility of formalizing all contentful arithmetic of whole positive numbers in such systems as Frege's and Russell's. Frege, surely, did not know this. He thought that he had succeeded in formalizing all of contentful arithmetic.
85. R. Carnap: 'Mathematik als Zweig der Logik' *Blätter für deutsche Philosophie* 4 (1930) 299.
86. On the question of the relations of mathematics and logic, see the article of A. D. Getmanova in the collective work *Logičeskie Issledovanija* [Logical Investigations], USSR Academy of Sciences, 1959.
87. *Grundgesetze I*, p. 1.
88. Th. Ziehen: *Lehrbuch der Logik auf positivistischer Grundlage mit Berücksichtigung der Geschichte der Logik*, Bonn, 1920, pp. 182, 232.
89. H. Scholz: *Geschichte der Logik*, Berlin 1931, p. 57. On page 4 of this work, Scholz says that Bolzano and Frege were the two greatest logicians of the nineteenth century.
90. *Translations from the Philosophical Writings of G. Frege*. Edited by P. Geach and M. Black, Oxford, 1952.

II

THE THEORY OF SENSE OF GOTTLOB FREGE

The notion of meaning plays an essential role in contemporary logic. Bound up with it are the so-called *intensional* logical calculi, i.e. calculi in which the *principle of extensionality* does not work. Calculi of this sort aim at clarifying, by formal means, the meaning of expressions of a certain kind of ordinary contentful language (for example, the calculi of *strict implication* try to explain the meaning of expressions such as "*A* follows from *B*", "from *A* logically results *B*"). At the same time the notion of meaning is also required in the exposition of systems of classical two-valued mathematical logic. In this way proceed, e.g., Hermes and Scholz [23] and A. Church [22]. In this case the notion of the meaning of linguistical expressions is used in order to make in a natural way the transition from current languages, with which authors usually begin in the prefaces, to formalized ones. Here it is shown that the idea of meaning, naturally originated in the study of current languages, is superfluous in calculi similar to those which are described by the above authors, in virtue of which these calculi acquire an extensional character (*extensional logic*).

It is known that the notion of meaning was first submitted to a
503 careful analysis in the works of the eminent German logician and mathematician, Gottlob Frege (1848–1925).[1] It is interesting to see what moved Frege to introduce this notion in his logical theory and the role it plays in his logic.[2]

The idea of *name* is one of the most important logical notions. This notion has been employed for a long time not only in the current language – from which it was taken by science –, not only in linguistics, but also in logic. Frege made precise the notion of name in such a way that, conserving its affinity to the use of the word "name" in ordinary language, it came to be one of the essential concepts of mathematical logic. This notion – together with "object",

"function", "idea", and some others – had a central place in the logical interest of the prominent German mathematician.

Frege rejected the traditional division of names into singular and general, but introduced the distinction between names of objects and names of functions. This was related to what Frege meant by object, namely: that which is not a function. In particular, Frege considered concepts as particular cases of functions; concepts, from this point of view, are such functions that to every argument is correlated either truth or falsity.[3] Because of this, expressions designating concepts (properties), and also relations, are for Frege a particular case of names of functions. Names designating objects are called by Frege *proper names*, names designating functions *functional names*, names designating concepts (properties) *concept-words*.[4]

Proper names are names of single, individual objects.[4] Examples 504
of such names are:

(*1*) "Aristotle",
(*2*) "Venus",
(*3*) "Master of Alexander the Great and disciple of Plato",
(*4*) "Morning Star",
(*5*) "Whoever discovered the elliptical form of planetary orbits",
(*6*) "Evening Star",

etc.[5]

Frege gives the following explanation of what he means by *proper name*: "by 'sign' and 'name' I have here understood any designation representing a proper name, which thus has as its reference a definite object (this word taken in the widest range), but not a concept or a relation . . . The designation of a single object can also consist of several words or other signs. For brevity, let every such designation be called a proper name" [5], p. 27.

To every proper name Frege relates, first, what he calls the *reference* (*Bedeutung*) of the name, and, secondly, what he calls the sense (*Sinn*) of the name. The reference of the name is that object which is designated (named) by the name. The sense of a proper name, for Frege, may be described as that knowledge, that information, which is included in the name, and the comprehension of the name

by a man, i.e. the assimilation of this information (see [22] § 01).

Concerning the expressions "sense" and "reference" Frege fixes the following way of speaking:

505 "A proper name (word, sign, sign-combination, expression) *expresses* its sense, *stands for* or *designates* its reference. By means of a sign we express its sense and designate its reference" ([5] p. 31).

In his first important work, *Begriffsschrift*, published in 1879, Frege did not yet make the distinction between the sense and the reference of a name. In *Grundlagen der Arithmetik* published in 1884 [10], the notion of sense did not appear either. It was only introduced in 1892, in the well-known article of Frege 'Über Sinn und Bedeutung' [5]. The motive which brought Frege to distinguish names not only according to their references but according to the sense expressed by them, was the study of identity.

What is identity?[6] Is it a relation between objects or between names, signs of objects? This is the question which Frege asks in that article. In his *Begriffsschrift* Frege still held for the second solution to the question. The argumentation which he develops in the article 'Über Sinn und Bedeutung' is the following: "$a = a$ and $a = b$ are obviously statements of different cognitive value: $a = a$ holds *a priori* and, according to Kant, is to be labelled analytic, while statements of the form $a = b$ often contain very valuable extensions of our knowledge and cannot always be established *a priori*. The discovery that the rising sun is not new every morning, but always the same, was one of the most fertile astronomical discoveries. Even to-day the identification of a small planet or a comet is not always a matter of course. Now, if we were to regard identity as a relation between that which the names 'a' and 'b' designate, it would seem that $a = b$ could not differ from $a = a$ (i.e. provided $a = b$ is true). A relation would thereby be expressed of a thing to itself, and indeed one in which each thing stands to
506 itself but to no other thing"[7] ([5] p. 25–26).

In fact, if we adopt this point of view, then sentences:

(1) "The morning star is[8] the morning star"

and

(2) "The morning star is the evening star"

happen to express (under the condition that sentence (2) is true[9]

– and it is actually true) one and the same fact, namely that the planet Venus is identical to the planet Venus. Which is evident.

At the same time, it is clear that both sentences are completely different as concerns their cognitive value. One could hardly maintain that sentence (1) contains real knowledge. Sentence (2) has another character. It widens knowledge, reporting an interesting astronomical fact.

The apparent confusion may be explained by the fact that the object, which is identical to itself, remains undifferentiated under the names "a" and "b", "Morning star" and "Evening star", with the help of which that identity is stated.

Whence may be seen what the solution of the difficulty will be. Frege characterizes it as follows: "What is intended to be said by $a = b$ seems to be that the signs or names "a" and "b" designate the same thing, so that those signs themselves would be under discussion; a relation between them would be asserted. But this relation would hold between the names or signs only in so far as they named 507
or designated something. It would be mediated by the connection of each of the two signs with the same designated thing." ([5] p. 26).

Therefore, the equality $a = b$ is a relation said about the name a of a certain object and the name b of a certain object and consisting in the fact that the objects of both names are identical. There is a pair (the object, its name). Because of this it may be said (and Frege says this) that $a = b$ is a statement about a and b only in so far as they designate something.

Nevertheless, such a conclusion cannot satisfy Frege. Because the designation of an object with any sign depends on our convention, "Nobody can be forbidden to use any arbitrarily producible event or object as a sign for something. In that case the sentence $a = b$ would no longer refer to the subject matter, but only to its mode of designation; we would express no proper knowledge by its means." (*ibid.*)

It is an evident fact that sentences, about the meaning of which it may be said that it limits itself to expressing that the object, named "a", has also the name "b", do exist. Such for example is the sentence:

(3) "Cicero is Marcus Tullius".[10]

It may be considered that in this proposition some knowledge is included too, consisting in this: the man whose name is Cicero is called in another way Marcus Tullius. But such knowledge does not concern the objects themselves, but the signs with which we designate these objects.

But, are all sentences about identity like this? Are there not among them such that express knowledge in a proper sense? Do not sentence (2) and sentence

(4) "Aristotle is master of Alexander the Great and disciple of Plato"

belong to this group?

The proposed solution of the question obliges us to consider sentence (4) as completely analogous to sentence (3). This means that sentence (4) must be understood as affirming only one thing: the man who happens to have received the name "Aristotle" is identical with the man whom people call "Master of Alexander the
508 Great and disciple of Plato"; moreover, we do not have the right of considering this second name as an expression composed of meaningful parts ("master", "Alexander the Great", "master of Alexander the Great", etc.) and giving the information that the man designated by this name studied under Plato and taught Alexander, but we are obliged to consider it as a sign arbitrarily designating Aristotle and, of itself, not giving any other information. The error of such a treatment of this sentence is evident. There is no doubt that sentences (3) and (4) differ as to their cognitive value. Essentially, the proposed solution of the problem does not give the possibility of distinguishing sentences "$a = b$" and "$a = a$" from the point of view of their cognitive significance. If the sign "a" differs from sign "b" only as concerns the material aspect but not as a *sign*, i.e. not as to how it designates something, then there won't be an essential difference between the mentioned sentences when sentence "$a = b$" is true (the difference consisting in the fact that tokens appearing at both sides of the equality sign in the first sentence are similar, but are not in the second sentence, is not essential for knowledge).

A differentiation of the significance of sentences "$a = a$" and "$a = b$" and a revelation of the cognitive value of sentences of the

second type, seems possible if to each name is associated not only the object which is designated by this name (the reference of the name) but also the way in which the name designates the object: its *sense*.

Frege makes clear the notion of sense in the following way. Let *a*, *b* and *c* be straight lines uniting the vertices of a given triangle with the middle points of the opposite sides. Then the point of intersection of the straight lines *a* and *b* coincides with the point of intersection of the straight lines *b* and *c*. We have, therefore, different designations (names) for one and the same point and these names ("point of intersection of the straight lines *a* and *b*", "point of intersection of the straight lines *b* and *c*") show, Frege says, how the designatum is given to us. In fact, both names designate differently 509
one and the same object. The first designates it as the point of intersection of the straight lines *a* and *b*, but the second as the point of intersection of the straight lines *b* and *c*. In both cases the object is given to us differently. The difference in the way in which the object is given to us through its designations, is the difference between the information about the object which is contained in its names. Because of this, the sentence "the point of intersection of the straight lines *a* and *b* is the point of intersection of the straight lines *b* and *c*" supplies true knowledge.

"It is natural now to think of there being connected with a sign (name, combination of words, letter) besides that to which the sign refers, which may be called the reference of the sign, also what I should like to call the *sense* of the sign, wherein the mode of presentation is contained. In our example, accordingly, the reference of the expressions "the point of intersection of *a* and *b*" and "the point of intersection of *b* and *c*" would be the same, but not their senses. The reference of "evening star" would be the same as that of "morning star", but not their sense" (*ibid.* p. 26–27).

Frege distinguishes the associated idea (*Vorstellung*) from the reference and the sense of a sign. While the reference of a proper name is a sensibly perceived object, the idea of somebody about this object is an interior image, derived from recollections concerning the sensible impressions which the man had before. The idea (*Vorstellung*)

is subjective: it is frequently saturated with emotions; the clarity of its different parts is not the same and is always changing; even in one and the same man ideas associated with one and the same sense are different at different times; the idea of one person is not the idea of another. A painter, a horseman, a zoologist, says Frege, will probably associate with the name "Bucephalus" very different ideas.

The sense of a sign differs radically from the idea evoked by this sign, because the sense "may be common property of many and, therefore, is not a part or a mode of the individual mind; for one can
510 hardly deny that mankind has a common store of thoughts which is transmitted from one generation to another" ([5] p. 29).

Sense, for Frege, has an intermediate place between the reference of the name which is the object itself designated by this name, and the idea (*Vorstellung*), having a subjective character; although the sense is not the object itself, it is not possible to consider it as being subjective. Frege makes the following comparison. "Let us suppose that somebody observes the Moon through a telescope. I compare the Moon itself to the reference; it is the object of the observation, mediated by the real image projected by the object glass in the interior of the telescope, and by the retinal image of the observer. The former I compare to the sense, the latter is like the idea or experience. The optical image in the telescope is indeed one-sided and dependent upon the standpoint of observation; but it is still objective, inasmuch as it can be used by several observers. At any rate it could be arranged for several to use it simultaneously. But each one would have his own retinal image. On account of the diverse shapes of the observers' eyes, even a geometrical congruence could hardly be achieved, and an actual coincidence would be out of the question. This analogy might be developped still further, by assuming A's retinal image made visible to B; or A might also see his own retinal image in a mirror. In this way we might perhaps show how an idea (*Vorstellung*) can itself be taken as an object, but as such is not for the observer what it directly is for the person having the idea" ([5] p. 30).[11]

511 The relation between the name and the designated object (in a

given language) is commonly called to-day the *designation* or *imenovanija* (this is the translation of the English term "the name relation" in the Russian edition of R. Carnap [29]). This relation consists in the fact that one and the same object may have different names, but a given name may be the name of only one object. From Frege's point of view, the name-relation is mediated by the relation of the name to the sense and of the sense of the name to the reference. "The regular[12] connexion between a sign, its sense and its reference is of such a kind that to the sign there corresponds a definite sense and to this in turn a definite reference, while to a given reference (an object) there does not belong only a single sign" ([5] p. 27).

The sense of a name, in the theory of Frege, may be described as that information about the object expressed in the name (fixed in language), univocally characterizing the object (see [22] § 01).[13] To the sense of a given name, corresponds (or should correspond) a certain and unique object – the reference of the name; here it may be said (Church [22] § 01) that the sense of the name defines the reference of the name. At the same time, one and the same reference may be defined by different senses.

It is necessary to emphasize that the comprehension of the sense 512
of a name does not imply that its reference is absolutely known. Through the analysis of the sense of a name, it is not always possible to determine its reference. Because of this, Frege says: "Comprehensive knowledge of the reference would require us to be able to say immediately whether any given sense belongs to it. To such knowledge we never attain "([5] p. 27). For this reason it is possible to know the sense of a name and to know nothing about the object of the name, excepting that the object is determined by this sense; it may happen that we know the meanings of two names, but we do not know whether they determine one and the same object or not. Moreover, it may happen that the object determined by the sense of a given name does not exist at all. The sense of the name does not determine the existence of the object. Clarifying what he had said previously about the relation between sense and reference, Frege says: "It may perhaps be granted that every grammatically well-formed expression representing a proper name always has a sense. But this is not to say that to the sense there also corresponds a

reference. The words 'the celestial body most distant from the Earth' have a sense, but it is very doubtful if they also have a reference. The expression 'the least rapidly convergent series' has a sense but demonstrably has no reference, since for every given convergent series, another convergent, but less rapidly convergent, series can be found. In grasping a sense, one is not certainly assured of a reference" ([5] p. 28).

Names having a sense but not designating any object are not "authentic" proper names; they only play the role of proper names; Frege calls them *fictitious* proper names. Fictitious names are to be found in common languages, but in the building of science, says Frege, it is not allowed to make use of them. In a formalized language there must not be a place for such names. Frege writes: "A logistically perfect language (*Begriffsschrift*) should satisfy the conditions that every expression, grammatically well-constructed as a proper name out of signs already introduced, shall in fact designate an object, and that no new sign shall be introduced as a proper name without being secured a reference" ([5] p. 41).[14] And this is how
513 Frege built his *Begriffsschrift*. In the *Grundgesetze der Arithmetik* he says that for complete rigor the principle which says "every well-formed sign must designate something" is essential. ([3] p. XII). Following this principle Frege formulates special rules securing for every well-formed name of his system a certain fixed reference (now such rules are called *semantical*).

Concerning the relation between a *name* and its *sense*, Frege writes: "To every expression belonging to a complete totality of signs, there should certainly correspond a definite sense; but natural languages often do not satisfy this condition, and one must be content if the same word has the same sense in the same context" ([5] p. 27–28). One and the same name – not only in different but
514 even in the same language – may express a different sense. Expressions with many senses are a rather frequent phenomenon in natural languages.[15] This must be eliminated if the language is intended to serve logic. In this case each name must express only one sense (from which follows that it must have only one reference).

One and the same sense may be expressed by different names. Names, expressing the same sense, are called *synonyms*.[16] As far

as synonyms have one and the same sense, they also have an identical reference.

In ordinary languages proper names naturally are divided into simple and complex (composed). This division is also retained in the formalized languages. A complex name is one made up of meaningful parts: as such, may appear proper names as well as designations of concepts, logical connectives and other expressions. Complex names are also called descriptive names or descriptions. (*3*), (*4*), (*5*) and (*6*) may serve as examples of descriptions. A name entering into the constitution of another name is called a component name. For example the component names "Plato" and "Alexander the Great" enter into the constitution of the name (*3*). Not every complex name has 515
component names: so names (*4*), (*5*) and (*6*) do not contain component names.[17]

Simple names are not made up of meaningful elements. They may enter into the composition of other names, but they themselves do not contain names. As examples of simple names (otherwise called elementary) may serve (*1*) and (*2*).

A proper name arbitrarily designates a determined object. It is up to the man who gives the name to assign to the named object this or that sign, its name. A composed name does not designate an object arbitrarily, but in function of the sense of its parts. For example, the sense of the name (*3*) [18] depends on the sense of the proper names "Alexander the Great" and "Plato", but also on the sense of the concept-words "to be a teacher of somebody" and "to be a disciple of somebody". If we don't understand one of these expressions, then we will not understand the complex name either.[19] In the formation of the sense of the name (*3*) also participates the connective "and". 516
Although, taken by itself, it does not express any sense and does not designate any object, its role in the formation of the sense of complex names is essential (this is clear as soon as we exclude it from the name (*3*): it not only loses its previous sense, but in fact loses any meaning at all). The name (*3*) is constructed according to given rules (e.g. its component names are in the nominative case; the connective "and" is between the expressions "Master of Alexander the Great" and "disciple of Plato"). Violation of the rules, according to which

a name is constructed, can also destroy its sense. This means that not every sequence of meaningful expressions in a given language is a meaningful expression. In every language there exist rules of formation of meaningful expressions, including names. These rules belong to the grammar of the language. In ordinary languages they are never completely fixed. In practice, nevertheless, we distinguish – in most cases without effort – the meaningful from the meaningless expressions.[20] In addition to the grammatical rules, the total context and the situation also help.

In formalized languages the rules of formation of meaningful expressions must be rigorously formulated. The first to understand this and to achieve it was precisely Frege. In his *Begriffsschrift* [Frege's system in *Grundgesetze* and not Frege's book] the primitive names are enumerated, and the ways of forming names from other names and expressions of his language are exactly indicated. Thus is defined the notion of "correctly formed name"; according to this, every correctly formed name has not only a sense but also a reference.

Then, the sense of a complex name is determined by the sense of its parts and the character of the rules according to which it is constructed; its sense changes when the sense of any of its parts
517 changes. If we consider that the sense of each part is determined by its linguistic character and that the rules according to which the name is composed are fixed in the grammatical structure of the name, then it becomes clear that the sense of the name is expressed by means of language and only thus.

This is perfectly natural, because the sense of a name is the knowledge included in it about its reference. This must find for itself a material, linguistic expression, i.e. be given in the name itself, in the sign itself, in its form, in the construction and character of its parts. Otherwise, how do we grasp its sense, wherefrom do we receive the information about its reference?

However, not every peculiarity of linguistic constitution and structure of complex names serves for the expression of sense. For example instead of

(7) "Square root of 4 which is less than 0"

it is possible to say

(8) "negative square root of 4".

This means that these names have an identical sense (synonyms); the difference in their linguistic construction does not exercise, for Frege, any influence on their sense. However, Frege does not give any determined indication about how to distinguish those peculiarities of the composition and structure of names which serve for the expression of sense from those which are indifferent for the sense. This is connected with the fact that he does not establish criteria for the identity of senses (cf. [22] § 02, note 37). The problem of synonyms, which consists in finding a method, for a sufficiently great set of languages, allowing to decide the question of the identity of sense for any two names, is not considered by him. As we shall see below, this is not accidental. For Frege's purposes it is enough to say that senses may be distinguished and identified, employing the given languages in some way or another. He himself in his works distinguishes and identifies senses of concrete names, doing this in full accordance with the feeling – proper to anybody – for the mother tongue.[21]

Although Frege does not pose the problem of synonyms, the question itself arises from the whole content of his researches. 518

Let us consider the sense of elementary names. For a long time in logic the opinion has prevailed that such names as (*1*) and (*2*) do not express any sense. Thus thought, e.g., J. S. Mill [13]. He divided names into general and singular and, among the latter, distinguished the class of proper names. The proper names of Mill are elementary names, and singular names which are not proper are descriptive names. In Mill's theory, complex names have both reference and sense.[22] Simple names – "John", "London", "England" and others – although designating objects, do not have sense since they do not express any characteristics which would belong to this object. Mill says: "So, I call a certain man Sophroniscus, but I also call him by another name: the father of Socrates. One and the other name designate the same person; but their content is completely different: they apply to this person under different aspects: one, only in order to distinguish this man from other people about whom one speaks, the other with the purpose of declaring one fact concerning this man, namely this, that Socrates was his son" ([13] p. 31).

Mill is completely right in stressing the different cognitive function of simple names and descriptions. Frege, too, saw this very well. But he could not agree with Mill because, in that case, the concept of sense would lose its generality and thus it could not achieve the role assigned to it by Frege's theory. The principle "every proper name has a sense" is an inseparable part of Frege's system.

The question arises: in what consists the sense of simple names? What information do they give about the object? A. Church [22], having the purpose of adopting the theory of Frege, says that elementary names do communicate about the object that the object is
519 called by this name. And this is its sense. Knowledge about the name of the object is, after all, some knowledge, too.[23]

Nevertheless, Frege himself answered the question in another way. He said that in ordinary languages opinions about the sense of such names as "Aristotle" may differ. "It might, for instance", writes Frege, "be taken to be the following: the pupil of Plato and teacher of Alexander the Great. Anybody who does this will attach another sense to the sentence 'Aristotle was born in Stagira' than will a man who takes as the sense of the name: the teacher of Alexander the Great who was born in Stagira" ([5] p. 27, note).

It might be thought that Frege's view contradicts his theory's basic premiss according to which sense is the "common property of many people". But this is wrong. Frege starts from the fact that the use of simple names in current languages presupposes the existence of definitions of these names, connecting them one to the other. In definitions, both the *definiens* and the *definiendum* have not only the same reference but also the same sense. This approach is connected with the peculiarities of Frege's logical calculus. Every new simple name in it, excepting the eight undefined functional names, is introduced by means of a definition establishing that it must have the same sense and the same reference as the *definiens* (which consists of already known signs). The fact that in ordinary language generally admitted definitions of such names as "Aristotle" do not exist, does not affect the fact, according to Frege, that everybody who is using the name "Aristotle" must have in mind some one of its definitions, i.e. he must associate with it some sense; the fact

that different people will have in mind different senses does not lead to a misunderstanding, since they have in mind the same object.[24] 520
Fluctuations of sense are not admissible in the construction of science, and must not be found within a "perfect language".

The principle of substitution of names having the same reference is the most important thesis of Frege's theory.[25] It may be expressed as follows: *if one of the component names which enter into a given complex name is substituted by a name having the same reference, then the complex name obtained as result of such a substitution will have the same reference as the first given complex name.* Thus, if in the name (*3*) the component name "Plato" is substituted by the name "the founder of the Academy", having the same reference, then the reference of the complex name resulting from such a substitution

(*9*) "The Master of Alexander the Great and disciple of the founder of the Academy"

will be identical to the reference of the name (*3*) (both names refer to Aristotle).

Concerning the *sense* of a complex name, which we obtain after such a substitution, it is impossible to say anything definite; it may or may not be identical with the sense of the original name. In our example, names (*3*) and (*9*) have different senses. Now if we take name (*9*) and substitute it in the component name "the founder of the Academy" by the name "whoever founded the Academy" (which Frege would consider as synonym of the substituted one), then the resulting complex name must be considered as coinciding 521
in sense with name (*9*). Because of this Frege says that *the sense of complex names does not depend on the reference of the component names.*

For the purposes of his theory of sense and reference this was not directly necessary. It is possible to ask the question: which are the correlations for senses, as obtained from Frege's theory. The answer is as follows:

If one of the component names, occurring in a given complex name, is substituted by another name of the same sense (i.e. a synonym), then the newly obtained complex name expresses the same sense as the original one (see [22] § 01 and also § 02, note 30); in

virtue of the principle of substitution of names having the same reference, the resulting complex name will have a reference coinciding with the reference of the initial complex name.

In Frege's theory the following thesis is also legitimate: if one of the component names occurring in a given complex name is substituted by a name the sense of which is different, then the resulting new complex name expresses a sense other than that of the original name. Further, the reference of the resulting complex name – in virtue of the many-one character of the relation of sense to reference – may coincide with the reference of the initial name, but may be found too to be different from it.

There remain cases when the component name is substituted by a name which has a different reference. The resulting complex name may have a reference different from the reference of the original name as well as one coinciding with it.[26] The sense will always be different from the sense of the original name.

522 Frege never bases himself on the principle of senses. Because its application presupposes that the relation of identity is defined for senses (i.e. a definition of synonyms is given) and this definition is not contained in Frege's theory. Only the rule of substitution of names having the same reference is necessary in it.[27]

For languages where names not having reference are found, one principle is still applicable: *if the component name does not have reference, then the complex name, in the constitution of which this component name occurs, does not have reference either.* This principle may be read in this way too: if the complex name has reference, then every one of its component names has reference, too.

523 The point of Frege's theory consists in the fact that sentences are particular cases of names. Frege shows that according to a strict interpretation everything applicable to names happens to be true for sentences too.

Wherein lies the sense and the reference of sentences? For the discussion of this question Frege starts from the fact that every declarative sentence contains a *proposition,* which may only be either the reference of the sentence or its sense. If we suppose the first, then for sentences the principle of substitution of names

having the same reference is not applicable. In fact, if in the sentence

(5) "The morning star is a body illuminated by the sun"

the complex name (*4*) is substituted by the name (*6*), which has the same reference, we obtain the sentence

(6) "The evening star is a body illuminated by the sun",

which must express the same proposition as (*5*). Now, (*5*) and (*6*) contain different propositions. This may be seen at least from the fact that a man not knowing that the morning star is the evening star, would consider one proposition false and the other true, which could not happen if both sentences would serve for the expression of one and the same proposition. In this way the principle of substitution is violated. Which means, concludes Frege, that the proposition cannot be the reference of the sentence; on the contrary, it must be considered as the *sense*.

What happens now with the reference of sentences? First of all, Frege remarks that there are sentences not having reference. Among them belong those containing names not having reference; for example: "Odysseus was cast on the shores of Ithaca in a deep sleep." Let us note that to speak of the truth or falsity of this sentence is not meaningful. If Odysseus did not exist, then what meaning has it to ask the question *whether it is true or false* that he was cast on the shores of Ithaca in a deep sleep?[28] Whence comes the presupposition that truth and falsity should be considered as the reference of 524
sentences. The following consideration, according to Frege, gives to this presupposition complete persuasiveness.

A predicate, says Frege, is attributed or rejected not in relation to the name, but in relation to its *reference*, i.e. the *object* itself.[29] Only when there is an attribution or denial of the predicates in relation to the *object* of the name does the sentence become true or false.[30] If the name which is the subject does not have reference, then the sentence is neither true nor false. But in this case (according to the principle which says that a complex name does not have reference if any component name occurring in it does not have reference) it cannot have reference either. Further, if as Frege says very rightly, the predicate is attributed or denied in relation to the object itself and not to its name, then the truth (or falsity) of the sentence remains unchanged, of course, when we change the names designat-

ting the object of the judgment, provided that these names will designate one and the same object. As a result, if we see truth and falsity as the reference of sentences, then the principle according to which if a complex name has reference then each of its component names has reference too, as well as the principle of substitution of names having the same reference, are valid for sentences too.

From all of Frege's doctrine on names it follows that the sense of the sentences, i.e. the proposition expressed in them, is determined only by the sense of its parts, and not by their references. For the
525 understanding of the proposition it is not necessary to know whether the component names have reference or not. In fact, we usually try to explain the reference of the component names. This means that we recognize a reference to the sentence itself. "But now why do we want every proper name to have not only a sense but also a reference?", asks Frege, "Why is the proposition not enough for us? Because, and to the extent that, we are concerned with its truth-value" ([5] p. 33). By *truth-value of a sentence* Frege means the fact that it is either true or false. *Truth or falsity* make up the reference of a declarative sentence.[31]

The notion of truth-value plays an important role in mathematical logic. Even before Frege's time the notion of truth-value was employed by the American logician, Peirce.[32] Frege employed the notion of truth-value for the first time in the paper *Funktion und Begriff*, dating from 1891 [9]. But only in the article *Über Sinn und Bedeutung* did he introduce this notion as an organic part of his theory of sense and reference.

Truth and falsity are considered by Frege as *objects*. It is known that such a treatment of truth and falsity is widespread in contemporary mathematical logic. For example, in the tabular construction of propositional calculus the functions of this calculus are generally considered as defined in the domain composed of two "objects": "falsity" and "truth", which in turn receive significance from this domain. Surely the transformation of *truth* and *falsity* into
526 *abstract objects* is only a way of simplifying the logical work. Frege, introducing the notions of truth and falsity as particular logical objects, seems not to have seen clearly enough that from the gnoseological point of view such an introduction is nothing more than a

certain *auxiliary procedure*, the efficacity of which is precisely conditioned by its limited character. From the dialectical materialist point of view the interpretation of truth and falsity as abstract objects is derived* from the notion of truth as the correspondence of our knowledge to material reality and from the dialectical materialist notion of objective truth as that content of representations and thoughts which does not depend on man or on mankind, but is a reflection of exterior reality.

From the general principles of the Fregean theory it follows that if a sentence has a truth-value, this is determined by the *proposition* contained in the sentence. Now, the proposition for Frege does not have a subjective character. It is, says the German logician, the objective content of thinking, which may be common property of many people.

All sentences, according to Frege's theory, are divided into two classes: the class of sentences the sense of which is such that it determines the truth-value "truth" and the class of sentences the sense of which is such that it determines the truth-value "falsity". The proposition is the sense of the names of truth or falsity.[33] A true sentence is the name of *truth* and a false sentence is the name of *falsity*. We may understand the sense expressed in a sentence, but not know which is the truth-value determined by it. It may even happen that the sentence is neither true nor false (this is the case when names not having reference enter into the composition of sentences) but that we, although understanding the sense, are not aware of it.

The following question arises at this point. In enunciating sen- 527
tences, people usually want to express not simply that the propositions contained in them are either true or false, but they intend the truth of the statement. But the affirmation of truth is not at all contained in the sentences as names of truth or falsity. In order to eliminate this difficulty, Frege introduces the notion of judgment (*Urteil*).

Frege calls judgment "the acknowledgment of the truth of a sentence" ([3] p. 9). To the extent that the sentence is considered merely as a name for truth or falsity, there is in it not yet any affirmation. The latter will occur only when to the sentence will be annexed an indication about its truth. In ordinary thinking and

* Here the Russian text is not quite clear.

scientific knowledge the expression of any sentence is normally regarded as the assertion of the truth; the assertion of the truth in this case is expressed by the very enunciation of the sentence.

In accordance with this, Frege considers it necessary to introduce in his *Begriffsschrift* a special sign for assertion. Frege writes that in the simple identity "$2^2 = 4$" there is no assertion; this identity merely indicates a certain (without knowing which) truth-value. In order to show that it is a question of the affirmation of truth, Frege puts before the name of a truth-value the sign " $\vdash$", so that in " $\vdash 2^2 = 4$" *it is asserted* that the square of two is four. "For me", says the author of *Grundgesetze der Arithmetik*, "assertion is not a useless cover of the proposition, but the acknowledgment of its truth-value" ([5] p. 34 note).[34]

The principle of substitution of names having the same reference, applied to sentences, concerns the truth-value. Frege writes: "If our supposition that the reference of a sentence is its truth-value is correct, the latter must remain unchanged when a part of the sentence is replaced by an expression having the same reference" ([5] p. 35).

The applicability of this principle to the most simple cases is evident. Thus, if in the sentence (5) the name (*4*) is substituted by the name (*6*), differing in sense but identical in reference, then the truth-value of the resulting sentence (6) coincides with the truth-value of the initial sentence (both are true).

Now let us consider this example:

(7) "George IV once asked whether Walter Scott was the author of Waverley."[35].

Let us substitute in (7) the name "author of Waverley" by the name "Walter Scott", which has the same reference. We obtain the sentence:

(8) "George IV asked whether Walter Scott was Walter Scott."

While sentence (7) is true (that the English king asked such a question is a historical fact), sentence (8) in all probability is false (it is doubtful that George IV wondered whether Walter Scott was Walter Scott). Thus we see that the principle of substitution is

violated. This is due to the fact that the substituting expression is inserted in reported speech.

As we shall see, Frege's theory eliminates paradoxes of this kind by explaining in an adequate way the sense and the reference of words in indirect discourse. We turn now to the examination of the Fregean interpretation of indirect speech.

529

The use of language is based on the principle according to which, if we want to say something in a sentence about any object, we don't use this object itself but its name ([20] p. 68 and 224; [21] p. 95). This principle is evident and cannot be violated, as long as the linguistic expressions themselves do not appear as object of the statement. Natural languages, employed for the purpose of communication of people among themselves, being the material covers of thought and instruments of the knowledge of the external world, are precisely distinguished by the fact that in them it is possible to speak about the language itself, about the expressions of the language and about the sense of the expressions of the language. In this case, if we confound the sign and the designatum there may arise a paradox, an example of which is cited by the translator of the book of S. K. Kleene *Introduction to Metamathematics*, A. S. Esenin-Vol'pin ([20] p. 225).

Frege knew this. He demanded a rigorous distinction between the object and its name, and criticized those who confuse the designator and the designatum (see [6]). In order to prevent such confusion when discourse is about the expressions of the language themselves, Frege either employed metalinguistic signs or put the linguistic expressions in quotation marks (beginning with [5] he started putting the expressions which were mentioned between commas). The fact that Frege payed such serious attention to the necessity of distinguishing the sign from the object designated by it, is his incontestable merit.

Frege distinguished (a) *mention* of the name (word, linguistic formation), (b) *direct use* of the name, and (c) *indirect* use of it.

Direct use of names is the customary use of them: in this case the name expresses its sense and designates its object. Sentences, in the direct use, express the proposition and – if it has a truth-value –

designate the truth or falsity. Mention of words takes place in the transmission of direct speech (quotation).[36] The words of the one who transmits the speech of somebody else have as their reference the words of this other man. So, in the sentence "Seneca wrote: '*Rationale animal est homo*' ", one is not talking about the proposi-
530 tion of Seneca nor whether this proposition is true, but only about the sequence of letters he wrote (example of Church [22] § 04, note 72). The sentence, " "*Rationale animal est homo*" ", has as reference the sentence "*Rationale animal est homo*". Frege writes: "If words are used in the ordinary way, what one intends to speak of is their reference. It can also happen, however, that one wishes to talk about the words themselves or their sense. This happens, for instance, when the words of another are quoted. One's own words then first designate words of the other speaker, and only the latter have their usual reference. We then have signs of signs. In writing, the words are in this case enclosed in quotation marks. Accordingly a word standing between quotation marks must not be taken as having its ordinary reference" ([5] p. 28).

Frege further showed that we may speak not only about the objects and not only about the names and the sentences, but also about the sense of the names and sentences. Which linguistic means are used in discourse about sense? If we want to speak about the sense of the expression "A", this may be done by means of the device "the sense of the expression "A" ". It is possible to speak about the sense in the form of reported speech. For example the sentence

(9) "NN understands what the center of gravity of the solar system is"

is an expression about the sense of the name "center of gravity of the solar system". The latter occurs in the context of reported speech. In reported speech words do not have their customary reference, and thus designate that which in the case of direct use is their sense. They are used, as Frege says, indirectly, or have an
531 indirect reference. Accordingly, Frege distinguishes the customary reference of a name and the indirect reference, and its customary sense and its indirect sense. The indirect sense of the name "A" coincides with the sense of the words "sense of the name "A" ".

Indirect reference of the name is its customary sense.

Every name in reported speech has indirect sense and indirect reference. Because of this a subordinate clause, expressing reported speech, has indirect sense and indirect reference; its reference is that which in its customary (direct) use would be considered as its proposition, and its sense coincides with the sense of the words: "the proposition that . . ." (where follows the corresponding sentence).

In the case of the indirect use of names the principle of substitution of names having the same reference cannot be applied in the way in which it is applied for names in their direct use; otherwise, from true premisses could be derived a false conclusion. This happened, too, in the example considered on page 528. From Frege's point of view the error here is due to the fact that the name "author of Waverley" in sentence (7) was used indirectly (because occurring in reported speech). This means that the reference of this name was its customary sense. Because of this, in applying the principle of substitution, we would be obliged to change this name for an expression having not the same customary, but the same indirect reference (for example, for the name "whoever wrote Waverley"). If sentence (7) is true, then the sentence obtained by such a substitution will be true also.

Whence the conclusion: the principle of substitution of names having the same reference allows the substitution of a name indirectly used only for a name having the same indirect reference.

This qualification also applies when a whole sentence undergoes the substitution. So, in the complex sentence "Copernicus thought that the visible movement of the sun was caused by the real movement of the earth", the subordinate clause (which – if it is considered separately, i.e. outside of the whole sentence in which it occurs – has the truth-value "truth") cannot be of course replaced by any sentence whatever expressing a true proposition; the substitution may 532
be done only with a sentence expressing exactly the same proposition as the substituted subordinate sentence.

For the indirect use of names (in particular of sentences) there is also a qualification for the principle which says that names (in particular, sentences) have no reference if none of their component names (in particular, sentences) have reference. From what was said above,

it clearly follows that if a component name has an indirect reference, the fact that this name lacks a direct reference does not mean, nevertheless, that the complex name as a whole does not have such. In the sentence "In the Odyssey it is said that Odysseus was cast on the shores of Ithaca in a deep sleep" the name "Odysseus", although not having a customary reference, has actually an indirect reference, which is the sole relevant reference in the given example. Because of this, the sentence as a whole has a truth-value; precisely, it designates the truth.[33]

533 The question arises: what is the criterion which Frege sees, the use of which enables one to decide that a given name is used directly, indirectly or belongs to some quoted direct speech? Apparently, Frege considers the forms of the given language as this criterion. The analysis which he makes in [5] is completely oriented by the forms of German. A sign of the indirect use of an expression is for Frege the fact that it occurs in the composition of a reported speech, but the reported speech is expressed by a subordinate clause. Frege classes subordinate clauses of German, in which is expressed reported speech, as a special category of subordinate clauses, which he calls abstract noun-clauses (*Nennsatz*); such sentences are introduced by the connective "*daß*". Reported speech – in German – is expressed in the subjunctive. Such are the signs indicating the indirect use of words. A tendency to analyse the forms of ordinary language for the solution of the questions connected with sense and reference, is characteristic for Frege. At the same time he realized clearly the difference between the linguistic and the logical approaches to language.

But in the real human languages it is not always easy to distinguish the cases when the speech is about the sense of the name from those when the statement concerns the object of the name. For example, in the complex name "the myth of Pegasus"[38] the component name "Pegasus" must be considered as having an indirect reference, even though here there is no reported speech. In fact, the word "Pegasus" designates here the sense which was given to this name by Greek mythology; this is suggested by the words "myth of". This is why "myth of Pegasus" has a reference, even though in it occurs a name not designating any object. In this way a name

may have an indirect reference and not occur in reported speech.

We know that if a name has an indirect reference, it is not possible to apply to it in the ordinary way, either the principle of substitution of names having the same reference, or the principle which says that a complex name has no reference if any of the names occurring in it does not have a reference. This allows us to consider a violation of one 534
of these principles – since their application is valid for direct use of names – in regard to any name, as a sign of the indirect use of the name. Frege actually has this criterion.[39] Its application to the previous example easily allows us to establish the fact of the indirect use of the word "Pegasus".

The principle of substitution of names having the same reference must be right, too, when the expression to be substituted is itself a sentence. "Now if our view is correct, the truth-value of a sentence containing another as part must remain unchanged when the part is replaced by another sentence having the same truth-value" ([5] p. 36). But, if the substituted sentence is a subordinate clause, then there are cases when the substitution of such a sentence by a sentence with the same truth-value generates falsity from truth. In order to clear up such cases and show *how* the principle of substitution of names having the same reference should be applied to subordinate clauses, Frege distinguishes different types of subordinate sentences, dividing them into four groups. The Fregean analysis concerns the forms of German, but it is not difficult to find a Russian counterpart to these forms.

The *first group* consists of sentences expressing reported speech.[40] Frege gives the following examples:

(10) "Copernicus thought that *the orbits of the planets are circular*",

(11) "Copernicus thought that *the visible movement of the sun was caused by the real movement of the earth*".

In both italicized subordinate clauses words are used indirectly. 535
Because of this, each sentence, taken as a whole, has an indirect reference: it is the proposition expressed in the sentence but not the truth-value. In such cases it is not possible to render the sense of the subordinate clause in an independent sentence. In fact, the sentence

"the orbits of the planets are circular" expresses another sense than the same sentence in a subordinate function. The sense of the subordinate one may only be given by means of a description: "the proposition that the orbits of the planets are circular". Because of this, subordinate sentences of the mentioned kind may be regarded as proper names of those propositions which are expressed by the corresponding sentences, if taken separately. The principal sentence together with the subordinate has as its sense one and only one proposition, the parts of which are not themselves propositions; the sense of the subordinate sentence is part of this single proposition. Because of this, the truth of the complex sentence does not include in itself either the truth or the falsity of the subordinate sentence. This is evident in examples. In (10) the subordinate sentence (if it is considered as an independent sentence) expresses a false proposition, but the sentence (10) as a whole is true; in (11) the subordinate sentence expresses a true proposition, and the complex sentence is true. This is completely understandable. Because the proposition here is not the sense of the subordinate sentence, but its reference; this is why, for the truth of the whole, it is indifferent whether this sentence is true or false (because the reference of the sentence is defined only by the reference of its parts and not by their sense). Such a subordinate sentence may not be substituted by a sentence having the same truth-value, but may be substituted only by one having the same indirect reference (expressing the same proposition). Only in such a way can the rule of substitution be applied in this case.

The *second group* of subordinate sentences consists of such sentences as (sometimes together with a part of the principal sentence) serve for the formation of complex names of objects. As an example Frege considers the following sentence:

536 (12) "*Whoever*[41] *discovered the elliptical form of planetary orbits* died in poverty"

The subordinate sentence entering in (12) is a name of *Kepler*. The words in it have a direct reference. Subordinate sentences of this group have in common the fact that they contain so called *indefinitely indicating expressions*, which also make possible the connection between the subordinate and the principal sentence. In our

example such an expression is "Whoever". In mathematics and mathematical logic to the indefinitely indicating expressions correspond the variables, bound by operators (e.g. the operator of description and the universal quantifier).[42] The indefinitely indicating expressions do not have reference and do not express any complete sense.

The subordinate sentences of the second group do not express complete propositions and do not designate truth or falsity. The sense of sentences of this kind may not be expressed in a single non-subordinate sentence. The application of the principle of substitution of names having the same reference to this sentence means the substitution of it by another name of the same object.

The sentence occurring in the following complex sentence is an example of a subordinate clause of the *third group*:

(13) "Napoleon, *who realized the danger threatening his right flank*, himself led his guard in the attack on the enemy positions."[43]

In sentences of this kind words have the customary sense and the 537
customary reference. The subordinate sentence expresses a complete proposition and its reference is truth or falsity. The proposition expressed by the entire complex sentence is made up of the proposition of the principal sentence and the proposition of the subordinate sentence. In the given example both propositions are conjunctively united. Therefore, the reference of the complex sentence is defined by the truth-values of the sentences conjunctively associated. Since the subordinate sentence has a customary sense and a customary reference, it may be substituted by a sentence having the same truth-value.[44] Thus, in sentences of this kind, the principle of substitution is applied in its inmediate form.

A more complicated situation arises when the subordinate sentence – because of a connection with another sentence – expresses more than it does alone. Sometimes in such cases words in the subordinate sentence are taken in two ways: once in the direct reference and once in the indirect reference. This happens in reported speech after such words as "to imagine", "to lie", etc. For example, in the sentence:

(14) "*A* lied [when he said] that *he saw B*"

are expressed two propositions, for which it would be inexact to say

that one of them belongs to the principal sentence and the other to the subordinate sentence. These propositions are:

(a) *A* affirmed that he saw *B*,

(b) *A* did not see *B*.

Expressing the first proposition, words of the subordinate sentence have an indirect reference, while the words expressing the second proposition have a direct reference. Therefore, the subordinate sentence occurring in (14) may not be substituted by a sentence having the same truth-value.

538 The indicated substitution also becomes impossible sometimes, in cases when words in the subordinate sentence are taken only in direct reference. Precisely this happens when the subordinate sentence expresses not only a complete proposition but also a part of another proposition. This is the case, e.g., in the sentence:

(15) "*Since the specific weight of ice is less than that of water,* ice floats in water."

We have here three propositions:

(a) The specific weight of ice is less than that of water,

(b) if something has a specific weight less than that of water, it floats in water,

(c) ice floats in water.

The subordinate sentence occurring in (15) expresses not only the first proposition, but also in part the second one. Therefore, it may not be simply substituted by another sentence having the same truth-value, because if we proceeded in this way, we would modify the second proposition, too; the modification of the latter could also affect its defined truth-value which, again, could affect the truth-value of the whole proposition (15).

Now, how should the principle of substitution of names having the same reference be applied in complex sentences containing a subordinate sentence of the *fourth group*? The complex sentence should first be submitted to logical analysis revealing the propositions contained in it; this means to substitute the given sentence by another sentence coinciding in sense with it but in which all the propositions are clearly presented.

For example, as a result of such an analysis the sentence (14) would appear thus:

(16) "*A* asserted that he saw *B*; and *A* did not see *B*".

The applicability of the principle of substitution of names having the same reference to the sentences "*A* asserted that he saw *B*" and "*A* did not see *B*", considered each as a whole, is evident. Concerning the subordinate sentence "he saw *B*", the rule of substitution is also suitable, but only in the form in which this rule is applied to reported speech.

Frege understood that not every sentence easily undergoes analysis in the way proposed by him. He wrote: "An examination of all 539
the subordinate clauses which one may encounter will soon provide some which do not fit well into these categories (in question are Frege's four groups of subordinate clauses). The reason, so far as I can see, is that these subordinate clauses have no such simple sense. Almost always, it seems, we connect with the main propositions, expressed by us, subsidiary propositions which, although not expressed, are associated with our words, in accordance with psychological laws, by the hearer" ([5] p. 46). In such cases, it should be exactly explained what the man had in mind in uttering the given sentence, and only after this to make the analysis of the latter.

Mathematical logic – at least in its fundamental, "classic" part including the ordinary two-valued propositional calculus, and the predicate calculus – has an *extensional* character. In it is valid the so-called *principle of extensionality*, according to which two *predicates* (*properties* or *relations*) do not differ if they have the same *extension*. This principle received a precise formulation after Frege introduced into logic the notion of predicates as *logical functions*, i.e. as functions relating to the objects (couples, triples, etc. of objects) of the assumed objective domain, the truth-values truth or falsity.[45] In extensional logic a predicate is considered as given if its extension is indicated, i.e. it is said in any way to which objects (a couple, a triple, etc.) of the considered objective domain the predicate relates 540
"truth". Therefore, the possibility arises of simply identifying properties with sets of objects, and relations with sets of couples, sets of triples, etc. of objects. Properties and relations, considered in this way, may be called properties and relations in the extensional sense. In mathematics the extensional approach is completely justified. It

is well known that the instruments of extensional, set-theoretical logic are sufficient for the foundation of most of contemporary mathematics.

Naturally the question arises: of what type was the logical calculus built by Frege, by means of which the prominent German logician undertook the foundation of arithmetic ([3] and [4]); was the thesis of extensionality at work in the Fregean calculus? The Fregean calculus possessed an extensional character. If two predicates (two logical functions) $\phi(x)$ and $\psi(x)$ for any argument take the same value, then we may, asserts Frege, transform the universality of this equation into an identity of the extensions which correspond to these predicates. "This possibility must be regarded as a logical law, which we, however, have always used, although tacitly, when we have spoken about the extensions of concepts. In it is founded by and large the Leibnizian-Boolean logical calculus" ([3] p. 14). Here it has to be particularly stressed that the extension of a concept (i.e. the class of objects for which a given logical function takes the value "truth")[46] is considered by Frege as a special *logical* object (similar to both truth-values).

It is known that another form of the principle of extensionality is the *Leibnizian axiom of identity*. It says that two objects are identical if, and only if, everything which is true of one object is true also of the other, and conversely. This axiom may also be expressed in the form of a rule (let us call it the *Leibnizian rule*)

$$\left\downarrow \frac{p = q}{\phi(p) \equiv \phi(q),} \right\uparrow$$

which is read thus: if p is identical to q, then in any sentence ϕ
541 containing p, the latter may be substituted (in all or in some places of the sentences ϕ where p occurs) by q and, therein, the truth of the assertion does not change; conversely, if such a substitution of p by q is possible for any sentence ϕ, then p is identical to q ([21] pp. 91–92 and the note of the editor on page 293).[47]

Since extensions are objects, they fall under this rule. This means that if for any class A something is said, then this may also be said for a class B if A coincides with B. But "speaking" about the class A may be done not only by means of the expression "class A" but also

using the notion of the property which defines the given class. For the class of men something may be asserted not only by using the expression "humanity" but also by employing the concept *man* (i.e. the concept concerning the property *to be a man*). It is known that one and the same class may be determined by different properties. From Leibniz' rule it follows that concepts about properties and relations, determining one and the same class, i.e. equiextensional concepts, may be substituted one for the other. For example, the concepts *straight line joining a vertex of an equilateral triangle to the center of the opposite side* and *straight line bisecting an angle of an equilateral triangle*, are equiextensional. Therefore, according to the rule of Leibniz, they may be substituted one for the other in any sentence. Thus, from the true sentence:

(17) "The straight lines joining the vertices of an equilateral triangle with the centers of the opposite sides intersect in one point" is obtained the true sentence:

(18) "The straight lines bisecting the angles of an equilateral triangle intersect in one point". 542

Equiextensional concepts do not differ one from the other precisely in the sense that they are reciprocally substitutable in any sentence of a given science.[48] In this – and only in this – sense, concepts having the same extension are identified; in this – and only in this – sense, it may be said that concepts to which corresponds one and the same class of objects may be identified with this class.[49] Nevertheless, it is not difficult to find contexts in which the substitution of equiextensional concepts one for the other will give, from the true, the false. Thus if it is true that

(19) "*NN* knows that the straight lines joining the vertices of an equilateral triangle with the centers of the opposite sides intersect in one point",

then, from this does not follow at all the truth of the sentence:

(20) "*NN* knows that the straight lines bisecting the angles of an equilateral triangle intersect in one point".

Really, if the sentence (19) is right, this does not guarantee at all the truth of the sentence (20), because, *NN* knowing that about which the first sentence speaks, may not know at all that about which the second sentence speaks. We see then, that there are special contexts

– in Quine's words "opaque" – in which the rule of Leibniz fails.

Frege formulated a principle which in a sense is more general than the Leibnizian rule: the rule of substitution of names having the same reference. Frege's rule concerns the substitution of expressions occurring in complex names. The introduction of the notion of truth-value and the idea of sentences as names of truth and falsity,
543 made Leibniz' rule a particular case of Frege's. Frege's rule coincides with Leibniz' when the substituted name forms part of a sentence.

We know that in Frege's theory the rule of substitution of names having the same reference works for all contexts without exception; to see how the rule applies, it is only necessary that one makes a previous and correct logical analysis and interpretation of the given expression. This rule also works for the "opaque" – or, as they are sometimes called, "intensional" – contexts, which we have considered above; and it, so to speak, clarifies them.

In (19) and (20) we had reported speech, and expressions in reported speech have indirect reference. Therefore, we are not right in interpreting the expressions:

(*10*) "the straight lines joining the vertices of an equilateral triangle with the centers of the opposite sides",

and

(*11*) "the straight lines bisecting the angles of an equilateral triangle"

as having the same reference, because in the given context they do not designate the extension of the concept, but the sense, i.e. that which may be called property in a non-extensional sense.[50]

We also apply the Fregean principle of substitution to properties in the non-extensional sense. The expression (*10*) in the constitution of the sentence (19) may be substituted by an expression designating the same property in a non-extensional sense, for example, by the expression "the medians of an equilateral triangle". (It is presupposed that, by definition, the median is a straight line joining the vertex of a triangle with the center of its opposite side.)

544 We said that the notion of *sense* was introduced by Frege in order to explain sentences containing *identity*. But the role of the Fregean notion of sense goes beyond this question. In fact, the role of sense in

his theory consists in giving extensional character not only to the logical calculus which Frege builds especially for the foundations of arithmetic, but also to ordinary thought and ordinary language, as far as the latter is used for the purposes of logic. Frege's theory of sense embraces the ordinary and the formalized languages.

Considering ordinary language, Frege met "opaque" contexts, in which it is as if the principle of extensionality would fail and in which expressions* would reveal, so to speak, that they have meaning. The best example of such contexts was indirect speech. This was a real difficulty, which precisely was first discovered by Frege.[51] For its solution there were two ways. The first way was to acknowledge that the thesis of extensionality really is violated in certain contexts of ordinary language and that the sense of expressions which is thereby revealed may be investigated by means of the construction of special logical calculi. This is intensional logic.

The second way was to defend the principle of extensionality. And 545
Frege did this. In order to defend the thesis of extensionality, Frege had to prove the universality of the rule of substitution of names having the same reference, and to do this was only possible by adequately interpreting the "opaque" contexts. And this, too, was done by the author of *Grundgesetze der Arithmetik*. He applied this procedure of logical analysis to such contexts, using his notion of sense. This notion gave to Frege the possibility of distinguishing the direct and the indirect use of words and *ipso facto* shed light on the logical nature of reported speech. Frege's explanation of reported speech is an essential step forward in the logical analysis of natural languages.

But on what is based the explanation of "opaque" contexts which Frege undertook? On the transformation of the senses of names and (considered as particular cases of them) of propositions, expressed by sentences, into a special kind of abstract object. For example the sentence (7) in relation to the name "author of Waverley" is an "opaque" context. According to the interpretation of reported speech presupposed by Frege, it is necessary to consider that in the given context this name designates its customary sense, which here is the reference of the name, i.e. an *object*; in accordance with

* The rest of this sentence is not quite clear in the Russian text.

such an interpretation, the rule of substitution should also be applied. The applicability of this rule shows that the context has an extensional character. Analogous is the situation for *propositions*. The sentence "The orbits of the planets are circular" in the context of sentence (10) has as reference a proposition, which is thus considered as an *object*.[52]

Now it is clear why Frege does not formulate a parallel rule for senses. Because when an expression reveals its sense, this becomes an object and falls under the scope of the rule of substitution of names having the same reference.[53] Frege operates with senses in
546 an extensional way.

A purely extensional interpretation of logical thinking will, it seems, always come up against difficulties. These suggest that such an interpretation has its limits. Extensional logic is a certain formalization of the logic of real thinking and, as every formalization, leads to the fossilization of the object, in the given case of real human thinking. This fossilization appeared clearly enough in the theory of Frege, particularly in the Fregean interpretation of conditional sentences. Let us consider this more in detail.

From the sense of a sentence – the proposition expressed in it – Frege distinguished what he called the "light", the "colour" of the proposition. For example, the use of the connective "although" gives to the sentence a peculiar light, which may come to be completely inconvenient if we substitute the subordinate sentence introduced by this connective by another sentence with the same truth-value.[54] Nevertheless, from a logical point of view, Frege considered this substitution as admissible, as if the light of the sentence did not concern logic.

According to Frege the situation in reference to the conditional sentence is analogous. The idea of the logical – according to the meaning – entailment of the consequent of a conditional sentence by its antecedent would be only the "colour" of the sentence, i.e. an *accessory* idea which, properly speaking, is not at all expressed by a sentence with the connective "if-then". Let us hear Frege himself.
547 He examines the sentence "if the sun has already risen, the sky is very cloudy". "Here", writes Frege, "it can be said that a relation

between the truth-values of conditional and dependent clauses has been asserted, viz. such that the case does not occur in which the antecedent stands for the True and the consequent for the False. Accordingly our sentence is true if the sun has not yet risen, whether the sky is very cloudy or not, and also if the sun has risen and the sky is very cloudy. Since only truth-values are here in question, each component clause can be replaced by another of the same truth-value without changing the truth-value of the whole. To be sure, the light in which the subject then appears would usually be unsuitable; the proposition might easily seem distorted; but this has nothing to do with its truth-value. One must always take care not to clash with the subsidiary propositions which are, however, not explicitly expressed and, therefore, should not be reckoned in the sense. Hence, also, no account need be taken of their truth-value'' ([5] pp. 45–46). In a note to this passage, Frege observes that the proposition contained in the considered sentence could also be expressed in the following way: "Either the sun has not yet risen, or the sky is very cloudy".

From the quoted words of Frege, it is evident that he interprets the conditional sentence purely extensionally, considering that *material implication*[55] is a *complete* formalization of the logical nature of such sentences. Frege's mistake is evident. It is known that for the formalization of mathematical proofs it is not necessary to show the logical consequence according to the meaning, but it is enough to use material implication. Making mathematical proofs, we are above all interested in obtaining from true premisses a true conclusion. For these purposes it is possible to use a logical apparatus with 548
material implication. But, from a logical point of view material implication is deficient because it badly corresponds to contentful thinking. Not all that people say, when they use the form of conditional sentences, can be formalized in the form of material implication. Whence the tendency to create a logical apparatus reflecting in the best way the laws of contentful thinking, and in particular, the meaning of logical entailment (B logically follows from A according to the content). The calculus of strict implication (Lewis, Ackermann) arose in fact from this tendency.

One should note that Frege did not see the problem of explaining

the sense of logical consequence, the expression of which is conditional judgment, i.e. the problem of strict implication.

The fundamental idea running through Frege's logical theory is the metaphysical conception of logic and its laws as something absolute and unchanging. From Frege's point of view the laws of thought are always and everywhere the same. If at some time thinking beings different from men were discovered, they too, he says, would think according to these laws.[56] Therefrom automatically followed the conclusion that logical calculi expressing these laws are essentially the same and must be similar to his *Begriffsschrift*. The *Begriffsschrift* which Frege built for the foundation of arithmetic had an extensional character. Possibly because of this, Frege considered as evident that the logic of the real and contentful thinking must also be extensional. In order to demonstrate the extensional character of ordinary and contentful logic, Frege introduced the notion of sense. He solved the problem[57] which he had set himself, but in doing so he completely eliminated from the domain of logic the "light" and the
549 "colour" of propositions, and even the notion of logical consequence according to content and meaning. The metaphysical character of Frege's conception impoverished his logic.

Frege did not understand that the construction – once and for all – of a law-endowed system of logic, valid for any language, is impossible. "Opaque" contexts are not definitely eliminable from contentful language because they are the expression of that indeterminacy which has its place in logic as in any other domain. The process of knowledge in logic consists in particular in the elimination of this indeterminacy and in the making precise of meaning and content. It is accomplished in a process of evolution. There is a succession of logical systems or, let us say, the systems themselves, so to speak, are in movement (they swell). In each stage of the process, in each concrete logical system there is a formalization of our knowledges referring to some concrete domain of objective reality[58]; in this, not every peculiarity of the contentful thinking, applied for the understanding of this domain of reality, becomes precise and clear. The construction of the subsequent logical systems, permitting a deeper formalization of the content and a

clarification of what had not been done before, leads as a rule to a more complete formalization of contentful thinking, but this formalization at every stage remains "confused", i.e. there is an un- 550
defined, unclarified, imprecise remainder.

Real human thinking is unlimited in its content, because this content is extracted from the objective world, which is inexhaustible; but it is unlimited, too, as to its structure or form (an expression of which are the diverse natural and artificial languages) because the latter works as an image of the unlimitedly rich content and has an evolution parallel to that of the content. The process of knowledge is always an asymptotic process. On this subject, Engels wrote: "Just as the infinity of knowable matter is composed of the purely finite things, so the infinity of thought which knows the absolute is composed of an infinite number of finite human minds, working side by side and successively at this infinite knowledge, committing practical and theoretical blunders, setting out from erroneous, one-sided, and false premisses, pursuing false, tortuous, and uncertain paths, and often not even finding what is right when they run their noses against it . . . The cognition of the infinite . . . therefore . . . from its very nature can only take place in an infinite asymptotic progress" ([2] p. 186). In logic, this progress consists in particular in a more and more complete revelation, by means of the formal apparatus, of the peculiarities of contentful human thinking, which is achieved in the course of the construction by logicians of always new logical calculi. In each stage of the evolution of logical symbolism we have some knowledge on the peculiarities of contentful thinking, and we attribute objective truth to this question, and this truth becomes, in the process of the evolution of logical systems, more and more complete and profound, remaining, nevertheless, relative in each stage. This means that to knowledge of the laws of the contentful thinking of people (as to the knowledge of any other subject) are fully applicable the theses of dialectical materialism about absolute and relative truth, for the explanation of which Lenin wrote in *Materialism and Empirio-Criticism*: " . . . Human thought then by 551
its nature is capable of giving, and does give, absolute truth, which is compounded of a sum-total of relative truths. Each step in the development of science adds new grains to the sum of absolute

truth, but the limits of the truth of each scientific proposition are relative, now expanding, now shrinking with the growth of knowledge'' ([1] p. 122).

But let us go back to Frege. The scientific creation of this very prominent German logician and mathematician offers itself as a good illustration of the above-quoted words of Engels about men who, making their contribution to human knowledge of the world, nevertheless start from frequently unfounded and one-sided premisses and commit theoretical blunders. The unfounded and one-sided premisses from which Frege started – the prejudice about the unchangeability of the objects of the world, the idea of the possibility of the diversification and identification of any object with which we have a cognitive concern, the conviction about the full universality and unchangeable character of the laws of thought considered by formal logic, the notion of the completely extensional character of the logic of contentful thinking, etc. – were *metaphysical* presuppositions. Frege did not see either the dialectical character of the process of knowledge as a whole, or how the dialectic appears in the evolution of mathematical logic itself.

But, leaving aside the metaphysical character of Frege's conception as a whole, his works (in particular his investigations about the notion of sense) have been a great contribution in the development of logic. Three lines in logic start with Frege. One of them consists in the construction of an extensional logic, oriented towards the foundations of mathematics. This line starts from the Fregean *Grundgesetze der Arithmetik*, which is the first fruit of the application of the apparatus of mathematical logic to the formalization of a concrete mathematical discipline. This line was continued by the authors of *Principia Mathematica* and other scholars dealing with the foundations of mathematics. The second line is more
552 related to logic in the proper sense and, although it is not directly outlined in the works of Frege, it arises from problems posed in his works. It consists in the construction of calculi elucidating the meaning of the contentful logical concepts, an example of which may be the calculi of Lewis and Ackermann [31], formalizing the notion of logical consequence according to the meaning, in the study

of logical modalities, subjunctive conditional sentences, sentences expressing the laws of science, causal relations, etc. The third line in logic finding its origin in the works of Frege is immediately connected with what has been the object of our article. It led to the semantic problems of contemporary logic. In fact, Frege's analysis of the notion of sense gave rise to a whole circle of problems of mathematical logic connected with the investigation of the relation between name and object, between designator and designatum. Along with Peirce, Frege was the founder of the domain of contemporary logic which afterwards received the name of logical semantics.[59]

In the past few years the question connected with the sense of linguistic expressions received great attention in relation to the 553
application of mathematical-logical methods for the investigation of the natural languages. The application of these methods developped together with the works on machine translation. A presupposition of automatic translation is an exact description of the relations between the expressions of the language from which the translation is done and those of the language into which the translation is done; only under this condition is the construction of an algorithm of translation possible. But, practice has showed that the grammatical description of languages which has been elaborated by ordinary linguistics is, as a whole, not well suited for the construction of such algorithms. Therefore, there is a question as to the more exact formulation of the grammar of natural languages. Thus arose the tendency in the research of language, which is now more and more called *mathematical linguistics*.

The fundamental problem of translation, including automatic translation, is the problem of the conservation of the meaning of an expression (of a given language) in the translation of it into another language. Already Frege, explaining the notion of the sense of a linguistic expression, characterized it as that which remains unchanged in the translation from one language into another. Therefore, it is natural that in mathematical linguistics those questions related to *sense* acquired a great importance. As we know, Frege did not formulate any rule concerning sense. Further evolution of semantics led to the construction of semantic systems in which the notion of sense was made more precise (see, e.g. [29]). The rise of mathe-

matical linguistics, devoted to the mathematical-logical analysis of natural languages, showed that, on the one hand, such systems can be useful in the practice of linguistic analysis but, on the other hand, revealed a strong limitation of many semantical systems (including those developped in [29], see the introduction of S. A. Janovskaja to the Russian edition of this book). The problems of semantics and, in particular, the questions connected with the sense of linguistic expressions, are nowadays hotly discussed in linguistic and mathematical-logical works, and it would be too soon, perhaps, to make predictions on the results which will impose themselves in this domain of research. Nevertheless, it is impossible to doubt that the
554 final explanation in this domain of logical and linguistic problematic will be supplied by *practice*. In this case, practice consists in the works on mechanical translation and the efforts for the creation of artificial information-languages and language-intermediaries used in automatic translation. And, it is evident that every success in this domain will be a step forward in the solution of that problem which Frege expressed with the words "try to see one and the same sense in its different garments" ([7] p. 196), stressing its importance for logic.

REFERENCES

1. In the consideration of the notion of sense the Stoics were Frege's predecessors. They were, it seems, the first in the history of logical thought to introduce a distinction between the *designator*, the word, the *object* or *objects* which the word indicates, and the content or *meaning* of the word – the *thought* expressed in it (see [11]).
2. I express my gratitude to Prof. S. A. Janovskaja, who suggested that I study the history of the notion of *meaning* in logic and whose advice helped me in the preparation of the present article.
3. For Frege's understanding of *function* and *concept*, see [32].
4. About names which do not designate any object, we shall speak below.
5. From the quoted examples, it appears that the Fregean use of the expression "proper name" differs from its use in everyday language. Expressions like (*3*) and (*5*) are not counted among names in ordinary language. Concerning expressions (*4*) and (*6*), there may perhaps be disagreements. This is understandable. Natural languages were not created for logical analysis, but emerged as means of communication among people, expressing their thoughts and sentiments. Logic must clarify its expressions. The Fregean explanation of the notion of name is an effort in this direction.

6. The word "equality" is used by Frege in the sense of identity, taking "$a = b$" just as "a is the same as b" or "b and a coincide". See [15] p. 25 and [3] p. IX.
7. Here Frege proceeds as a metaphysician. Identity of an object to itself is in no way a universal property. The development of contemporary science and, in particular, of mathematical logic – in full agreement with the philosophy of dialectical materialism – has made it evident that not every object may be described as identical to itself. But the discussion of this question falls outside the limits of our article.
8. According to Frege's theory in these and similar sentences the word "is" works not only as a copula but helps also in the expression of identity, i.e. is a part of the predicate. In sentences of the form "Aristotle is a philosopher", the word "is" appears only as a copula. (See [7]).
9. The truth of sentence (1) follows from its analytical character.
10. The example is not Frege's, but it is frequently employed by contemporary authors in discussions about meaning.
11. The author of the commentary to the Russian edition of *Tractatus Logico-Philosophicus* of L. Wittgenstein, V. K. Finn, maintains that "Frege, considering the sense of expressions of the real language, arrived at the conclusion that sense in language is subjective" ([28] p. 105). As the reader may see from our exposition of Frege's theory and from his own views, quoted in the present article, this assertion can hardly be accepted. Surely, the treatment of the notion of sense in Wittgenstein differs, in a series of points, from the Fregean conception of sense. We cannot here go into detail on this difference; we just remark that this difference does *not* consist in the fact that sense, in Frege's theory, would be in some way subjective while Wittgenstein assigns to it an objective character. We think that, if the author of the *Tractatus Logico-Philosophicus* (who is known to be a subjective idealist, in his philosophical conceptions) pretends to assign an objective character to sense, this may rather be explained because the notion of sense itself was borrowed by him from Frege.
12. The restriction "regular" is necessary because in natural languages (differing in this from most logical calculi) divergences occur from the relation formulated by Frege: for example, "names" occur not having reference. See below.
13. This information is based on the real properties of objects. From the dialectical-materialist point of view, senses of names are reflections of things of the objective world.
14. From the practice of mathematics, it is known that expressions not having a reference frequently are used in mathematical demonstrations as auxiliary means. Thus, certain theorems about real numbers may be proved – operating, during the proof, with the name "$\sqrt{-1}$", which designates nothing (in the domain of real numbers). But such expressions can always be eliminated, and the proof of a theorem containing "fictitious" or "imaginary" names may be substituted by another proof of the same theorem (which, it is true, may happen to be clumsier) in which "fictitious" names do not occur. In this sense, expressions not having reference are not necessary, in as far as it is possible to work

without them. But it would be false to veto the use, in mathematics, and mathematical logic, of names not having a reference and to exclude any use of them at all. The use of names not having a reference cannot do any harm if we realize, in the course of the proof, the fiction of the corresponding expressions and we exclude these names from the final results of the proof; at the same time, the use of such names frequently is convenient since it simplifies the reasoning.

It seems that fictitious names may not appear in sentences which are the result of a proof, precisely because the presence of such names makes them neither true nor false. This is the rational sense of the Fregean condition that in the building of science names not having a reference must not be used.

15. The fact of having many senses is not a weakness but rather a strong side of natural languages. It allows them to be more economical in the designation of a huge number of objects; it allows languages to constantly change under the influence of the requirements of life and practice, it allows better expression of the exact *nuances* of thought and of the experiences of people. As regards the misunderstandings which might arise because of expressions with many senses, they are averted by the fact that the sense of an expression becomes clear from the general context and from the situation in which it is used.
16. The use of the expression "synonym" in mathematical logic is different from the use of this word in grammar, where single (isolated) words are admitted as synonyms.
17. To such names of current languages, correspond in some formalized languages (in particular, in Frege's calculus) names formed by means of operators, the meaning of which may be rendered by the expression "that, which": compare the word "whoever" in the composition of (*5*), corresponding to the definite article in German and English.
18. In Russian it is impossible to see from the form of expression (*3*) that it is a proper name. It is different in German, where the definite article serves as characterizing sign of a proper name. That is why Frege assigns a great importance to the distinction between expressions with definite and indefinite articles. In Russian, in similar cases, it is not clear whether a given expression is a proper name or designates a concept. In order to show that expression (*3*) is a proper name, it would be necessary to express it by introducing in it the word "whoever": "whoever is the master of Alexander the Great and disciple of Plato". We don't do this in order not to complicate the exposition.
19. Everyone studying a foreign language is well acquainted with this situation. If one does not know the sense of a single expression, then he cannot know the whole in which this expression is found. (Sometimes we guess the sense of the whole, even if we don't know the sense of one of its parts. This is explained by considering that the whole context contains, in a series of cases, indirect indications about the sense of the given expression. But we never have a guarantee about the correctness of our conjecture. Because of this, it is convenient to consult a dictionary and find out the meaning of the unknown words [*sic*]. Only after this certainty is acquired about whether we grasped correctly the sense of the whole.

20. Because in ordinary languages rigorous criteria of distinction between meaningful expressions and expressions not having a sense are not available, contrary answers are frequently given to questions about the meaningfulness of an expression. The fact that living languages continually change should be taken into account too, because what yesterday was senseless today is found in the language and is a senseful expression.
21. For example, Frege says that the substitution of an adjective for a corresponding dependent clause or the substitution of a substantive for a corresponding dependent clause does not change the sense of an expression.
22. To the reference of complex names in the theory of Mill corresponds the so-called "direct reference" ("denotation", which the translator of Mill's works into Russian, V. N. Ivanovskij, gave with the term *označenie*, see [13]), and to the sense corresponds the *soputstvujuščoe značenie*, i.e. indications about the peculiarity of the object (connotation, translated by Ivanovskij as *sooznačenie*).
23. Without any doubt, knowledge about the *name* is different from knowledge about the *object itself*. But, for the sake of the generality of the theory, one may proceed as Church does.
24. Surely, it would be false to suppose that in ordinary languages there always exists a formal definition of the senses of such names as "Aristotle"; it would be more exact to say that in such languages proper names are vaguely defined, one by means of the other. Since it is generally known that objects, defined by the senses of appropriate names, do really exist, a circularity in the definition does not lead to lack of clarity or misunderstanding.
25. Frege says it is evident. In [5] he expresses it only for sentences (regarded as particular cases of names), which is clear from the content of this article. A more general formulation is to be found in [3]. We formulate the Fregean principle according to Church [22]. In contemporary works, this principle – and its variants and analogues in different semantical systems – is also called the *principle of interchangeability* of linguistic expressions. See [29] (for example, p. 95–97, 156–163); see also [3].
26. We give an example, where the reference of a complex name remains the same. Let us substitute in the name "capital of USSR" the complex name "USSR" by the name "RSFSR", the reference of which is certainly other than the reference of the substituted name. As a result we obtain the name "capital of RSFSR", the reference of which coincides with the reference of the primitive complex name.
27. The application of the principle of substitution to names having the same reference also presupposes, certainly, that the identity of references of names is somewhat defined. Acquaintance with Frege's works suggests that the German logician started from the idea that such a definition is always possible in virtue of men's cognitive capacity of identifying and distinguishing objects, but at the same time objects were described by him as something completely circumscribed, as rigorously different one from another, which in logic we have the right of treating as constant and unchangeable substances.

But this presupposition is too strong. In fact it is a certain idealization which we are justified in using in logic only as long as we don't forget that the evolution and movement of the objects of the objective world, their transformations and interactions makes this differentiation and identification practically unfeasible in many cases; the idea of the possibility of distinguishing and identifying objects is an abstraction from the real state of affairs. The thesis resulting from the doctrine of dialectical materialism, that human concepts always somehow simplify the objective connections of nature, only approximately reflecting it, artificially separating one or the other aspect of the process of development (see Lenin [1] p. 143) was, without any doubt, unknown to Frege, who was in no way a dialectician. As metaphysician, it seems, he said that every object in the world may always be distinguished and identified.

The character of the philosophical opinions of Frege is shown in [32]. About how Frege's metaphysical orientation is revealed in his theory of sense, see below.

28. Precisely because sentences containing fictitious proper names cannot be true, such names, says Frege, cannot be applied in science.
29. In Russia, M. I. Karinskij (in a sense) followed the same opinion (see [14] p. 398), but his view was connected with the materialist point of departure of his logic (see, e.g., [14] p. 387 and also [15]). In a similar way, Frege's view that the predicate is assigned or rejected not in relation to the name (logical subject) but in relation to the object of the name, is directly related to the materialist tendencies in his logical views. About Frege's views on logic, see [32].
30. This thesis also extends to cases where, in its turn, the object of the name is any name or, in general, any linguistic form, and also to the so-called *autonymous use of names*. In the autonymous use of linguistic forms, these appear at the same time as the *designatum*, the object itself, and as the *designator*, the name of this object. For the autonymous use of expressions, see [33].
31. Frege's theory of sense and, in particular, his way of understanding sentences had a deep influence on subsequent logic. But not all logicians admitted the Fregean treatment of sentences. Sometimes Frege's view is even called absurd (see [26]). But this accusation is unjustified. It is possible, of course, to be in disagreement with Frege. But one thing is beyond question: if the Fregean theory of the sense of names is accepted and one wishes to extend it to sentences, then, whether one likes it or not, one arrives at the description of truth and falsity as their reference.
32. See [22], *Introduction*, note 67.
33. It is necessary to keep in mind that Frege uses the expression "thought" [this would be the normal translation of *"Gedanke"*, to which, in this translation, "proposition" is preferred] in a special sense, as a term of his theory. While the term "thought" is usually applied not only to the sense of declarative sentences but also to the content (sense) of names (in the proper sense), of interrogative sentences, and of other expressions of language, Frege uses this term only in relation to the senses of declarative sentences (*Behauptungssätze*).

34. Frege's proposed distinction between the affirmation of a sentence and its use as a particular kind of name was later widespread in mathematical logic. Russell and Whitehead ([16] p. 92–93), I. I. Zegalkin ([17] p. 14–15; [18] p. 314–315, 321; [19] p. 207) and other authors, followed – with some exceptions, connected with the renunciation of certain Fregean semantical theses – the basic conceptions of Frege. Church ([22] 'Introduction') is perhaps the one who most fully accepts Frege's conception on this question.
35. This example is examined by Russell [24] and [25] (Chapter: 'Descriptions'), by Church [22] ('Introduction', § 01) and by other authors. The antinomy, making up the content of this famous example, is called in contemporary works the antinomy of the name relation (see [29], § 32).
36. The quotation of direct speech is not the unique form of mentioning words. Thus, in the following cases (in which there is no quotation of direct speech) the word "*maša*" [Mary] is mentioned:

    "'"*Maša*" consists of four letters",
    "the word *Maša* consists of four letters",
    "*Maša* consists of 4 letters".

    In the last example the name "*Maša*" is used autonymously – as the name of itself. In order that autonymy does not lead to paradoxes, it is necessary to have the possibility of always distinguishing the autonymous use of words from their non-autonymous use.
37. Concerning the application of the principle of substitution of names having the same reference to direct speech (and in general, to the mention of expressions), it is evident that the word (name, expression, complete sentence) may not be substituted by another word with the same customary reference or by a word with the same indirect reference as has the substituted name (expression, word, sentence). So, in the sentence:

    (A) "Seneca wrote: "Rationale animal est homo""

    the expression in quotation marks may be "substituted" only by the same expression. If one considers that the expression

    "Rationale animal est homo"

    (together with the quotation marks) is a name of the expression

    "Rationale animal est homo"

    and that, for the latter, it is always possible to form another name, for example, by using quotation marks of another kind

    "'Rationale animal est homo'"

    or say, by agreeing to designate it by the letter ϕ, then the application of the principle of substitution of names having the same reference gives the sentence

    "Seneca wrote: "'Rationale animal est homo'"

    and

    "Seneca wrote: ϕ".

    But the last sentence naturally appears as having already a completely different meaning than (A), which is seen in the tendency to consider "ϕ" in this sentence as an expression used *autonymously*.
38. The example is considered by Church in [22].
39. Its application presupposes that we can distinguish the use of a name (direct or indirect) from its mention.

40. But not every subordinate sentence, expressing reported speech, belongs to this group. See below.
41. This is how we translate the German definite article (ТОТ, КТО).
42. Another example of subordinate sentences containing indefinitely indicating expressions are the conditional sentences expressing universality, for example: "When the sun is at the tropic of Cancer, the northern hemisphere has its shortest day", and "if $x > 0$, then $x + 3x > 0$". In the first of these sentences, the undefined indication concerns time and is expressed by the present tense of the verb. In the second example, the indefinitely indicating part is the variable, x. Frege exposed his theory on variables in [8]. Contemporary mathematical logic, as concerns the understanding of variables, follows in general the way suggested by Frege.
43. We have in mind the battle of Waterloo.
44. Frege observes that, in relation to the sentences of type (13), it is necessary to introduce the following restriction: in the substituting sentence there must be the same subject as in the substituted one. The restriction loses its value if a connection is admitted by means of "and", considering sentence (13) as coinciding in sense and reference with the sentence "Napoleon realized the danger threatening his right flank, and Napoleon himself led his guard in the attack on the enemy positions".
45. The term "predicate" as used in contemporary logic (as synonymous with "logical function") is not used by Frege. He calls logical functions of one argument concepts (so that Fregean concepts coincide with general properties) and logical functions of two arguments relations.

    It is necessary to have in mind that in Frege's system every logical function is defined in the universal objective domain which includes any object whatever (Frege does not put any restriction on the expression "any"). This peculiarity of Frege's calculus caused its contradiction (a contradiction in Frege's system was discovered by Russell).
46. For the Fregean notion of the extension of a concept, see [32].
47. "p" and "q" are objective variables (in their place one may introduce names of objects of the objective domain), and "ϕ" is a variable for sentences; the sign "$=$" means identity of objects, and "$\equiv$" serves for the designation of the equivalence of sentences in relation to truth and falsity ("double implication"). The use of free variables for the writing of this rule expresses universality. The horizontal line is the sign of consequence, and the arrows indicate the consequence from the upper formula to the lower and from the lower to the upper.
48. It is supposed, naturally, that the logic of the given science is founded on the principle of extensionality.
49. Frege conceived the question precisely in this way. He did not introduce a *complete* identity of the concept with its extension. He did not say that the relation of identity happening between two equiextensional concepts eliminates every difference between them and transforms them into one concept. Replying to Kerry, Frege wrote: "If he believes that I am identifying the concept and the extension of the concept, then he is mistaken" ([7], p. 198). Apart from this, Frege generally took the extensional point of view.

50. Properties are understood in real thinking precisely in this way. *To be a straight line joining a vertex of an equilateral triangle to the center of the opposite side,* and *to be a straight line bisecting the angle of an equilateral triangle* are generally considered as different properties, even if, from the extensional point of view, they coincide.
51. It is interesting to note the following historical fact. It is known that the follower of the work of Boole and Morgan, the English logician, Jevons, put the so-called principle of substitution at the base of his theory. This principle was, as a matter of fact, the inaccurately formulated Fregean principle of substitution of names having the same reference (for the particular case when the substituted expression occurs in a sentence). Jevons gave to this principle a universal significance, maintaining that "every term occurring in any sentence whatsoever, may be substituted by a term about which it is asserted in any premiss that it is identical with the first" ([12] p. 48). He did not, it seems, see that his assertion was wrong in reference to sentences containing indirect speech. This could probably be explained by the fact that Jevons was completely unacquainted with the semantic aspects of logic. Contrary to the case of his compatriot, Mill, he did not understand that the study of the relation of linguistical expressions to the objects about which we speak with their help, has an essential significance for logic. Therefore, he overlooked intensional contexts to which his principle of substitution was not immediately applicable.
52. The applicability of the rule of substitution in the case of the mention of names in the quotation of direct speech, as it follows from note 37, is trivial, because quoted words are there the objects. Frege did not specially consider this case.
53. But in Frege's theory, it is certainly supposed that we can in some way distinguish and identify senses occurring as objects just as we do, according to Frege, with any object. See p. 517 and note 27, too.
54. Words in concessive subordinate clauses have direct reference.
55. From the extensional point of view, material implication is a set of couples (truth, truth), (falsity, truth), (falsity, falsity).
56. See Frege's introduction to his *Grundlagen der Arithmetik* [10].
57. It should be mentioned that Frege's way of putting into practice the extensional point of view in the ordinary languages is not effective because Frege's theory lacks rigorous criteria for dealing with the question whether a given expression is used directly or indirectly or mentioned. Neither purely linguistic criteria, about which we spoke on pages 532–533, and even less the method later mentioned on pages 533–534, are effective. Other known means of realizing the principle of extensionality in natural languages are also ineffective (for example, Russell in [25]). The effective realization of the principle of extensionality in ordinary languages is, as it seems, absolutely impossible, if only because living languages are in a state of continuous evolution; but one is possible for certain parts of natural languages, whose vocabularies and grammatical rules may be exactly defined by means of mathematical logic. This problem is also solved for the construction of the algorithms of mechanical translation from one language into another.

58. We have in mind interpreted logical calculi.
59. Mathematical logic understands the term "semantics" differently from ordinary linguistics. The former calls semantics a logical linguistic discipline, whose object of study is the relation between formally constructed calculi and the domain of reality which is reflected in it – its contentful interpretation; in semantics, one tries to make precise such notions, necessary in interpretation, like "name", "object", "sense" of linguistic expressions, "true", "false" in formalized languages, etc. Semantics as a scientific tendency, as intimately connected with the linguistic branch of mathematical logic, must not be identified with the trend in contemporary bourgeois philosophy, usually called "semantic philosophy", consisting in the idealistic interpretation and subjectivist distortion of the results of mathematical logic and logical semantics. But, it is necessary to observe that many works in the domain of semantics, which belong to foreign logic, reflect the subjectivist opinions of their authors. An example of this is the book of Carnap [29], a critical appreciation of which the reader will find in the introduction to the Russian edition by S. A. Janovskaja.

BIBLIOGRAPHY

[1] V. I. Lenin, *Materializm i empiriokriticizm. Sočinenija*, t. 14, Gospolitizdat, Moskva, 1947.
[2] F. Engels, *Dialektika prirody*, Gospolitizdat, Moskva, 1952.
[3] G. Frege, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, erster Band, Jena, 1893.
[4] G. Frege, *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*, zweiter Band, Jena, 1903.
[5] G. Frege, 'Über Sinn und Bedeutung', *Zeitschrift für Philosophie und philosophische Kritik* 100 (1892) 25–50.
[6] G. Frege, 'Kritische Beleuchtung einiger Punkte in E. Schröders Vorlesungen über die Algebra der Logik', *Archiv für systematische Philosophie* 1 (1895) 433–456.
[7] G. Frege, 'Über Begriff und Gegenstand', *Vierteljahrschrift für wissenschaftliche Philosophie* 16 (1892) 192–205.
[8] G. Frege, 'Was ist eine Funktion', *Festschrift Ludwig Boltzmann gewidmet zum sechzigsten Geburtstage 20. Februar 1904*, Leipzig, 1904.
[9] G. Frege, *Funktion und Begriff*, Jena, 1891.
[10] G. Frege, *The Foundations of Arithmetic*, Oxford, 1950.
[11] B. Mates, *Stoic Logic*, University of California Press, Berkeley–Los Angeles, 1953.
[12] S. Jevons, *Osnovy nauki* (translated by M. Antonovica), 1881.
[13] J. S. Mill, *Sistema logiki sillogističeskoj i induktivnoj*, 1914.
[14] 'Otryvok iz litografirovannogo izdanija M. I. Karinskogo "Logika" (1884–1885)', *Voprosy filosofii* 1947, 2, 387–396.
[15] P. S. Popov, 'O kurse logiki M. I. Karinskogo', *Voprosy filosofii* 1947, 2, 386–387.

[16] A. Whitehead and B. Russell, *Principia Mathematica*, Vol. I, Cambridge University Press, 1935.
[17] I. I. Zegalkin, 'O texnike vyčislenija predloženij v simvoličeskoj logike'. *Matem. sb.* 34 (1927) 1, 9–28.
[19] I. I. Zegalkin, 'Arifmetizacija simvoličeskoj logiki. (Prodolženie.),' *Matem. sb.* 36 (1929) 3/4, 205–338.
[18] I. I. Zegalkin, 'Arifmetizacija simvoličeskoj logiki. Teorija predloženij i funkcij odnogo argumenta', *Matem. sb.* 35 (1928) 3/4, 311–377.
[20] S. K. Kleene, *Vvedenie v metamatematike*, Moskva, 1957.
[21] A. Tarski, *Vvedenie v logiki i metodologiju deduktivnyx nauk*, Moskva, 1948.
[22] A. Church, *Introduction to Mathematical Logic*, Princeton, 1956.
[23] A. Hermes und H. Scholz, 'Matematische Logik', *Enzyklopedie der mathematischen Wissenschaften* Band I, Heft I, Teil 1.
[24] B. Russell, 'On Denoting', *Mind* (n.s.) 14 (1904) 479–493.
[25] B. Russell, *Einführung in die mathematischen Philosophie*, München, 1926.
[26] V. Valpola, *Über Namen. Eine logische Untersuchung*. Helsinki, 1950.
[27] W. O. Quine, 'Designation and Necessity', *Journal of Philosophy* 39 (1939) 701–709.
[28] L. Wittgenstein, *Logiko-filosofskij traktat*, Moskva, 1958.
[29] R. Carnap, *Značenie i neobxodimost'*, Moskva, 1959.
[30] C. S. Lewis, *Survey of Symbolic Logic*, University of California Press, Berkeley-Los Angeles, 1918.
[31] W. Ackermann, 'Begründung einer strengen Implikation', *Journal of Symbolic Logic* 21 (1956) 2, 113–128.
[32] B. V. Birjukov, 'O rabotax Frege po filosofskim voprosam matematiki', *Filosofskie voprosy estestvoznanija*, vyp. 2: *Nekotorye metodologičeskie voprosy fiziki, matematiki i ximii*, Moskva, 1959.
[33] B. V. Birjukov, 'Avtonimoe upotreblenie vyraženij', *Filosofskaja enciklopedija*, t. 1, Moskva, 1960.
[34] B. V. Birjukov, 'Vzaimozamenjaemosti otnosenie', *Filosofskaja enciklopedija*, t. 1, Moskva, 1960.

TRANSLITERATION TABLE

Russian Letter *Cap. Ital.*	Studies in Soviet Thought	U.S. Library of Congress	Russian Letter *Cap. Ital.*	Studies in Soviet Thought	U.S. Library of Congress
А *а*	a	a	Р *р*	r	r
Б *б*	b	b	С *с*	s	s
В *в*	v	v	Т *т*	t	t
Г *г*	g	g	У *у*	u	u
Д *д*	d	d	Ф *ф*	f	f
Е *е*	e	e	Х *х*	x	x͡h
Ё *ё*	ë	ë	Ц *ц*	c	t͡s
Ж *ж*	ž	z͡h	Ч *ч*	č	c͡h
З *з*	z	z	Ш *ш*	š	s͡h
И *и*	i	i	Щ *щ*	šč	s͡hch
Й *й*	j	ĭ	Ъ *ъ*	″	″
К *к*	k	k	Ы *ы*	y	y
Л *л*	l	l	Ь *ь*	′	′
М *м*	m	m	Э *э*	é	é
Н *н*	n	n	Ю *ю*	ju	i͡u
О *о*	o	o	Я *я*	ja	i͡a
П *п*	p	p			

INDEX OF NAMES

Ackermann, W. 85, 88, 99
Anscombe, G. E. M. XV, XVII
Aristotle XIII, 47
Asmus, V. F. VII
Axmanov, A. S. VII
Avenarius, R. 6
Beth, E. W. XVIII
Black, M. 51
Blakeley, T. J. XVII
Boethius X
Bolzano, B. 51
Boole, G. 42, 97
Brouwer, L. E. G. 1
Carnap, R. XV, XVI, XVIII, 45, 51, 59, 98, 99
Church, A. XIV, XV, 28, 50, 52, 59, 64, 72, 93
Comey, D. D. XVII
Dedekind, R. 3, 13, 47, 48
Dirichlet, 26, 50
Engels, F. 87, 88, 98
Erdmann, B. 5, 6, 9
Esenin-Vol'pin, A. S. 71
Finn, K. V. 91
Føllesdal, D. XVII
Geach, P. XV, XVII, 51
Getmanova, A. D. 51
Gödel, K. 2, 4, 51
Hegel, G. W. F. VIII, XXI
Heine 9
Hermes, A. 52, 99
Hilbert, D. 3, 11, 41, 46, 47, 48, 50
Husserl, E. XV, XVII, XVIII, 3, 46
Ivanovskij, V. N. 93
Jacoby, G. XVII
Janovskaja, S. A. 47, 90, 98
Jevons, W. 97, 98
Joseph, H. XXI
Kant, I. IX, X, XXI
Karinskij, M. I. 94, 98
Kerry 15, 96
Kleene, S. K. 71, 99
Kneale XV
Kol'man, E (A) VIII, 28, 50
Korselt 3
Kronecker, 3
Lalande XVIII
Leibniz, G. 45, 81, 82
Lenin, V. I. IX, XI, XIII, 6, 30, 48, 50, 87, 94, 98
Lewis, C. I. 85, 88
Liebmann, 15, 48
Lobačevskij 50
Lotze XVIII
Mach, E. 6
Mates, B. 98
Mill, J. S. XXI, 63, 64, 93, 97, 98
Misch XVIII
Morgan, A. de 97
Pavlov, V. T. XIV
Peano, G. 4, 42
Peirce, C. S. XXI, 4, 43, 68, 89
Petzoldt, J. 6
Plato VIII
Poincaré, H. 1, 47
Popov, P. S. VII, 98
Poreckij, P. S. 42
Quine, W. van Orman 82, 99
Rivetti-Barbó, F. XV
Rozental', M. M. XIV
Russell, B. VIII, XII, 2, 4, 16, 40, 41, 46, 48, 49, 50, 95, 96, 97, 99
Ruzavin, G. I. VII
Scholz, H. XV, XXI, 47, 51, 52
Schröder, E. XIV, 3, 4, 11, 34, 35, 42, 48, 50
Sluga, H. D. XIV, XVI
Steck, M. 48
Suarez, F. XVII
Tarski, A. 99
Tavanec, P. V. VII, XVII
Thomae 3, 12
Valpola, V. 99
Weierstrass, K. 47
Weyl, H. 1
Whitehead, A. N. 4, 95, 99
Wittgenstein, L. VIII, 91, 99
Wolff XXI
Zegalkin, I. I. 95, 99
Ziehen, Th. 47, 51.